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R. P. Boas, Jr., *Executive Editor*

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Mathematical Reviews

Vol. 6, No. 10 NOVEMBER, 1945 Pages 253-284

HISTORY

Marar, K. Mukunda, and Rajagopal, C. T. On the Hindu quadrature of the circle. J. Bombay Branch, Roy. Asiatic Soc. (N.S.) **20**, 65–82 (1944). [MF 12367]

This paper is a sequel to one by C. M. Whish [Trans. Roy. Asiatic Soc. **3**, 509–523 (1835)]. Its aim is to fill in details of the general results given by Whish. The paper is concerned with Hindu approximations to the value of π and takes as its starting point the series known in the Occident as Gregory's series. This is found in the Karaṇa-Paddhati and in the Sadratnamālā; the limiting case, for $t=1$, is proved geometrically in the Yukti-Bhāṣa. The authors give, from these śāstras, approximations derived from Gregory's series and rapidly convergent alternating series for π to which these approximations lead. In these alternating series, the nth term, as $n\to\infty$, is $O(n^{-2p-1})$ $(p=1, 2, 3)$ or $O(n^{-2p-2})$ $(p=1, 2)$. There is also a series of positive terms from the Karaṇa-Paddhati for $(\pi-3)/6$. The authors work out the method of the Yukti-Bhāṣa, which uses continued fractions, for transforming Gregory's series into these forms. There is a discussion of dates and authorship of the works cited.

An important part of the paper is an appendix by K. Balagangadharan which gives the Sanskrit ślokas and English translations of the main theorems used by Whish and by the authors and of other references of the paper. This appendix is of great interest to those who do not have the Sanskrit sources at their disposal.

[The numerator of the first term of the series for $(\pi-3)/6$ on page 73 should be 1 instead of 2.] *E. B. Allen.*

Gurjar, L. V. The value of $\sqrt{2}$ given in the Śulvasutras. J. Univ. Bombay (N.S.) **10**, part 5, 6–10 (1942). [MF 12186]

Gurjar, L. V. The problem of squaring the circle as solved in the Śulvasutras. J. Univ. Bombay (N.S.) **10**, part 5, 11–16 (1942). [MF 12187]

Schmidt, Olaf H. The computation of the length of daylight in Hindu astronomy. Isis **35**, 205–211 (1944). [MF 11942]

The problem mentioned in the title plays a great role in ancient astronomy because of its importance for the determination of geographical latitude. The methods used for its solution constitute an important test case for the relations between astronomical and mathematical doctrines in different regions and at different periods. In the present paper an astronomical Sanskrit table from the D. E. Smith Collection in New York is investigated. It is found that its computation follows rules which are found in the Siddhântas, based on spherical trigonometry but proceeding along lines essentially different from Ptolemy's methods.

O. Neugebauer (Providence, R. I.).

Thompson, J. Eric S. Maya arithmetic. Carnegie Institution of Washington publ. no. **528** (Contributions to American Anthropology and History no. 36), 37–62 (4 plates) (1942). [MF 12343]

Makemson, Maud Worcester. The astronomical tables of the Maya. Carnegie Institution of Washington publ. no. **546** (Contributions to American Anthropology and History no. 42), 185–221 (1943). [MF 12342]

***Sanchez Perez, Jose Augusto. Arithmetic in Babylonia and Egypt.** Consejo Superior de Investigaciones Cientificas, Madrid, 1943. 72 pp. (Spanish)

The Spanish title is La Aritmética en Babilonia y Egipto.

Burton, Harry Edwin. The optics of Euclid. J. Opt. Soc. Amer. **35**, 357–372 (1945). [MF 12469]

This is a literal translation of Euclid's Optics in its older version [ed. Heiberg, pp. 2–121]. No commentary is added; the figures are given as found in Heiberg's edition: that is, sometimes seriously distorted. *O. Neugebauer.*

Smyly, J. G. Square roots in Heron of Alexandria. Hermathena **63**, 18–26 (1944). [MF 12042]

The author discusses the method of approximation for all examples of square roots in Heron's "Geometrica" and "Stereometrica." *O. Neugebauer* (Providence, R. I.).

Bell, Eric Temple. Sixes and sevens. Scripta Math. **9**, 209–231 (1943). [MF 11940]

Bell, Eric Temple. Sixes and sevens. Scripta Math. **10**, 81–147 (1944). [MF 11938]

Bell, Eric Temple. Sixes and sevens. Scripta Math. **11**, 21–50 (1945). [MF 12776]

This paper gives a picturesque account of the early periods of Greek mathematics connected with the names of Pythagoras, Thales, etc. The author interweaves his narrative with numerous remarks of great generality, concerning what may be called philosophy of history; these do not always accord with the generally accepted interpretation of the facts.

O. Neugebauer (Providence, R. I.).

Miller, G. A. A tenth lesson in the history of mathematics. Nat. Math. Mag. **19**, 286–293 (1945). [MF 12476]

Boyer, Carl B. Early graphical solutions of polynomial equations. Scripta Math. **11**, 5–19 (1945). [MF 12775]

Boyer, Carl B. Historical stages in the definition of curves. Nat. Math. Mag. **19**, 294–310 (1945). [MF 12477]

Quaternion centenary celebration. Proc. Roy. Irish Acad. Sect. A. **50**, 69–75, 89–92 (2 plates) (1945). [MF 12194]

This issue contains an address by the President of the Academy, R. I. Best; messages from J. L. Synge and G. D. Birkhoff; an extract from a manuscript of Hamilton; and articles by E. T. Whittaker, A. J. McConnell, A. W. Conway, F. D. Murnaghan, and J. R. Colthurst [cf. these Rev. **6**, 199, and the two following titles].

McConnell, A. J. The Dublin mathematical school in the first half of the nineteenth century. Proc. Roy. Irish Acad. Sect. A. **50**, 75–88 (1945). [MF 12195]

Whittaker, E. T. The sequence of ideas in the discovery of quaternions. Proc. Roy. Irish Acad. Sect. A. **50**, 93–98 (1945). [MF 12218]

Dugas, René. Sur la pensée dynamique d'Hamilton: origines optiques et prolongements modernes. Revue Sci. (Rev. Rose Illus.) **79**, 15–23 (1941). [MF 12854]

Karpinski, Louis C. Copernicus, representative of Polish science and learning. Nat. Math. Mag. **19**, 343–348 (1945). [MF 12696]

***Kagan, V. F. Lobačevskij.** Scientific-Popular Series of Biographies. Akad. Nauk SSSR, Moscow-Leningrad, 1944. 347 pp. (Russian)

This book is more interesting from the historical than the mathematical point of view. It is shown by recently found church records that the commonly accepted birth date (November 3, 1793) is incorrect; these records show that Lobačevskij was born at Nižni-Novgorod on December 2, 1792. The first nine chapters set the scene; chapters 10 to 20 are concerned with Lobačevskij's mathematical work; this part gives a glimpse of Lobačevskij's attempts at topological treatment of geometry. The author credits Lobačevskij with the discovery of Graeffe's method, which appeared in Lobačevskij's little-known book on algebra.

The last four chapters are devoted to non-Euclidean geometry with its many applications to mechanics, physics and cosmology. The book concludes with a ten-page bibliography on non-Euclidean geometry and on foundations (up to 1937). The book is excellently done both from the mathematical and historical point of view; the only blemish is the unnecessarily harsh portraiture of Gauss, particularly in his relations toward the younger Bolyai and Abel.

M. S. Knebelman (Pullman, Wash.).

Hadamard, J. Obituary: Emile Picard. 1856–1941. Obit. Notices Roy. Soc. London **4**, 129–150 (1942). [MF 12509]

Whittaker, E. T. Obituary: Andrew Russell Forsyth. 1858–1942. Obit. Notices Roy. Soc. London **4**, 209–227 (1942). [MF 12511]

Weyl, Hermann. Obituary: David Hilbert. 1862–1943. Obit. Notices Roy. Soc. London **4**, 547–553 (1944). [MF 12461]

Hardy, G. H. Obituary: William Henry Young. 1863–1942. Obit. Notices Roy. Soc. London **4**, 307–323 (1943). [MF 12512]

Baker, H. F. Obituary: Geoffrey Thomas Bennett. 1868–1943. Obit. Notices Roy. Soc. London **4**, 597–615 (1944). [MF 12462]

Baker, H. F. Obituary: Geoffrey Thomas Bennett. J. London Math. Soc. **19**, 107–128 (1944). [MF 12558]

Hodge, W. V. D. Obituary: Tullio Levi-Civita. 1873–1941. Obit. Notices Roy. Soc. London **4**, 151–165 (1942). [MF 12510]

Burkill, J. C. Obituary: Henri Lebesgue. 1875–1941. Obit. Notices Roy. Soc. London **4**, 483–490 (1944). [MF 12460]

Fite, W. Benjamin. David Eugene Smith. Amer. Math. Monthly **52**, 237–238 (1945). [MF 12441]

Kosambi, D. D. Obituary: George David Birkhoff. 1884–1944. Math. Student **12**, 116–120 (1945). [MF 12494]

Obituary: George David Birkhoff. Bol. Soc. Mat. Mexicana **2**, nos. 1 and 2, 15–18 (1945). (Spanish) [MF 12869]

Rey Pastor, J. Professor George D. Birkhoff and his influence in Argentina. Revista Ci., Lima **47**, 105–109 (1945). (Spanish) [MF 12709]

Identical with an article in Revista Union Mat. Argentina **10**, 65–68 (1945); these Rev. **6**, 101.

Onicescu, Octave, et Vranceanu, Georges. La vie et l'oeuvre de Georges Ţiţeica. Bull. Math. Soc. Roumaine Sci. **43**, 147–161 (1941). [MF 12732]

Contains a list of the 92 papers by Ţiţeica published in volume 1 of his collected works (Bucarest, 1941).

NUMBER THEORY

***Čebyšev, P. L. Collected Works. Vol. I. Theory of Numbers.** Akad. Nauk SSSR, Moscow-Leningrad, 1944. 342 pp. (1 plate) (Russian)

[The Russian title is Polnoe Sobranie Sočinenij P. L. Čebyševa. Tom I. Teorija Cisel.] The first edition of the collected works of Čebyšev, which appeared in 1899 and 1907, contained all the memoirs and notes published during his lifetime with the exception of three dissertations, two of which had been published previously in book form while the third existed only in manuscript. This second edition is to contain not only the papers included in the first edition but also these three dissertations together with some unpublished material. The editors do not follow the chronological order of publication, but distribute the papers in five volumes according to their content. This first volume contains all Čebyšev's papers pertaining to the theory of numbers, including his well-known book, Teorija Sravnenij (Theory of Congruences). The influence which some of these papers exercised on the development of mathematics is well known and is stressed in two notes, one by A. Gelfond commenting on the famous memoir on prime numbers and another by B. Delaunay commenting on the series of investigations which grew out of the memoir Ob odnom arifmetičeskom voprose (On a certain arithmetical question).

J. V. Uspensky (Stanford University, Calif.).

Kaplansky, Irving. Lucas's tests for Mersenne numbers. Amer. Math. Monthly **52**, 188–190 (1945). [MF 12255]

The author gives a generalization of Lucas's tests for the primality of Mersenne numbers $2^{4n+1}-1$. Let $q=2^k-1$ and let c and d be integers chosen so that d and c^3-d are of the right forms to be quadratic nonresidues of q if q were a prime. Define a sequence of W's by $W_1=2c(c^3-d)^{-1}$, $W_{i+1}=W_i^2-2$. Then q is a prime if and only if q divides W_k. The case of $(c, d)=(1, 3)$ gives a sequence of W's beginning with $(-2)^1$, -4, 14, 194, $\cdots$ which is essentially one of Lucas's testing sequences. In case k is of the form $4n-1$ so that 5 would be a nonresidue of a prime $q=2^{4n-1}-1$, the case of $(c, d)=(1, 5)$ gives the sequence $(-1)^1$, -3, 7,

47, 2207, ···, the other famous Lucas sequence. The proof is short and self-contained. *D. H. Lehmer.*

Barker, Charles B. Proof that the Mersenne number M_{167} is composite. Bull. Amer. Math. Soc. **51**, 389 (1945). [MF 12517]

The author announces the result of his test of the Mersenne number $2^{167}-1$ by the famous Lucas method. Although the article does not specify which of the Lucas sequences was used, correspondence with the author reveals that it was the sequence 3, 7, 47, 2207, ···. Since the 166th term of this sequence was found to be not divisible by $2^{167}-1$, the latter number is composite. Another test of this number has been made by H. S. Uhler [Mathematical Tables and Other Aids to Computation **1**, 404 (1945)] with the same general result. Since the two computers used different sequences, the two final residues are not comparable.

D. H. Lehmer (Aberdeen Proving Ground, Md.).

Gangadharan, K. S. On some identities and a generalization of Wilson's theorem. Math. Student **12**, 75–77 (1945). [MF 12486]

Several identities similar to the following are derived:

$$\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}S_j=1/n,$$

where S_j is the sum of the reciprocals of $1, 2, \cdots, j$. The generalization of Wilson's theorem is that for a prime p, if $\lambda+r\equiv 0 \pmod p$ and $r\leqq p$, then

$$\frac{\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+p)}{\lambda+r}\cdot\binom{p}{r}\cdot\frac{1}{p}+(-1)^r\equiv 0 \pmod p.$$

I. Niven (West Lafayette, Ind.).

Kanold, Hans-Joachim. Folgerungen aus dem Vorkommen einer Gauss'schen Primzahl in der Primfaktorenzerlegung einer ungeraden vollkommenen Zahl. J. Reine Angew. Math. **186**, 25–29 (1944). [MF 12098]

If an odd perfect number exists, it must have the form

$$N=p^{\alpha}q_1^{2\beta_1}q_2^{2\beta_2}\cdots q_r^{2\beta_r}, \qquad \alpha \text{ odd}.$$

The author proves the following results. If p is a Gaussian prime, then $\alpha<r(r-1)$. A similar inequality for β_ρ is obtained if q_ρ is a Gaussian prime. Using these results the author proves that an odd perfect number must be divisible by at least one prime greater than 59; moreover, it must contain at least 5 different prime factors, and at least 9 different prime factors if $N\not\equiv 0 \pmod 3$. This improves results of Sylvester [Collected Mathematical Works, Vol. 4, Cambridge University Press, 1912, pp. 604–610].

A. Brauer (Chapel Hill, N. C.).

Gupta, Hansraj. On residue chains. Math. Student **11**, 54–55 (1943). [MF 12535]

The following theorem is proved. Let m be any integer for which a primitive root exists. Set $\varphi(m)=h$. There exists a chain $r_1, r_2, \cdots, r_{h-1}$ of the $h-1$ integers which satisfy $0<r_\nu\leqq m-2$ and $(r_\nu, m)=1$, such that the integers r_1r_2, r_2r_3, ···, $r_{h-1}r_1$ are congruent to the numbers $r_1, r_2, \cdots, r_{h-1}$ in a certain order. *A. Brauer* (Chapel Hill, N. C.).

Sagastume Berra, A. E. On the group of residues modulo n. An. Soc. Ci. Argentina **139**, 49–64 (1945). (Spanish) [MF 12593]

An expository paper dealing with the decomposition of the multiplicative group of the residues, modulo n, that are prime to n. Some applications are given; for example, it is proved that, if b is prime to n, then the exponent to which b belongs (mod n) is less than or equal to the order of the element $b-1$ in the additive group of all integers modulo n.

H. W. Brinkmann (Swarthmore, Pa.).

Overholtzer, Gordon. A new application of the Schur derivate. Bull. Amer. Math. Soc. **51**, 313–324 (1945). [MF 12274]

It is well known that the sum of the kth (k positive, negative, or zero) powers of the rational integers less than and prime to p^n (p a rational prime) is divisible by p^n if $p-1$ does not divide k or by p^{n-1} if $p-1$ divides k. The quotient of the division of such a sum by p^n is denoted by $S[n, x^k]$, which is thus a p-adic integer or has at most a denominator of p. This paper studies the p-adic properties of the sequence $\{S[n, x^k]\}$ and of related sequences. It is proved that this sequence is p-adically convergent and so are its Schur derivates of any order. The Schur derivate of a sequence of rational numbers $\{a_n\}$ with respect to p is defined to be the sequence $\{\Delta a_n\}$, where $\Delta a_n=(a_{n+1}-a_n)/p^{n+1}$. For positive k the p-adic limit of $S[n, x^k]$ is found to be

$$(-1)^{k/2-1}B_{k/2}(1-p^{k-1})$$

when k is even, and zero when k is odd (here $B_{k/2}$ is a Bernoulli number). The p-adic limit of $\Delta^m S[n, x^k]$ is evaluated in terms of the p-adic limit of $S[n, x^{k-m}]$. If $g(x)$ is a polynomial and $S[n, g(x)]$ is defined in the obvious manner, it is readily seen that similar results hold for the resulting sequence. The same is proved if $g(x)$ is a power series in x, if the valuation of the coefficient of x^i tends to zero as $i\to\infty$.

H. W. Brinkmann (Swarthmore, Pa.).

Täcklind, Sven. Über die Periodizität der Lösungen von Differenzenkongruenzen. Ark. Mat. Astr. Fys. **30A**, no. 22, 9 pp. (1944). [MF 12127]

Let $a_1, \cdots, a_m$ and k be given integers. It is well known that the elements x_n of a recurring series defined by

$$x_{n+m}+a_1x_{n+m-1}+\cdots+a_mx_m=0$$

are periodic mod k for every given set of initial elements $x_1, \cdots, x_m$. In this paper, the length l of this period is determined. If, for instance, $k=10$, $x_{n+2}-x_{n+1}-x_n=0$, and not all the elements are even, then $l=60$ if $x_{n+1}\not\equiv 3x_n \pmod 5$ for every n (series of Fibonacci); $l=12$ if $x_{n+1}\equiv 3x_n\not\equiv 0 \pmod 5$; $l=3$ if $x_n\equiv 0 \pmod 5$. If all the elements x_n are even, then $l=20$, 4, and 1, respectively, in these cases. The length of a period may be relatively long. For instance, the length of the period mod 10 of the series defined by $x_{n+7}-x_{n+6}-x_n=0$ is greater than 2,480,436 if at least one of the elements is relatively prime to 10. *A. Brauer.*

Shatrowsky, L. A new generalization of Davenport's-Pillai's theorem on the addition of residue classes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 315–317 (1944). [MF 12661]

H. Davenport proved [J. London Math. Soc. **10**, 30–32 (1935)] that, if $\alpha_1, \cdots, \alpha_u$ are different residue classes mod p and $\beta_1, \cdots, \beta_v$ are also different residue classes mod p, and if w is the number of distinct residues $\alpha_i+\beta_j$, then $w\geqq\min(p, u+v-1)$. This was generalized by I. Chowla [Quart. J. Math., Oxford Ser. **8**, 99–102 (1937)] and S. Pillai [Proc. Indian Acad. Sci., Sect. A. **6**, 179–180 (1937)]. The following generalization of these results is proved here. Let G be a group and let $a_1, \cdots, a_u$, $b_1, \cdots, b_v$ be two subsets, the b_j being permutable in pairs. Let w denote the number of

distinct elements a_i or $a_i b_j$. Then $w \geqq u$ and $w \geqq v$ (trivial), and $w \geqq \min(r, u+v)$, where r is the minimum order of the elements b_j. To deduce Davenport's result we can suppose $\beta_s = 0$, exclude it, and have p as the order under addition of each element $\beta_1, \cdots, \beta_{s-1}$. Applications are suggested to residue classes in algebraic fields and to lattice-point sets.

G. Pall (Montreal, Que.).

Rédei, L. Kurze Darstellung des fünften Gauss'schen Beweises für den quadratischen Reziprozitätssatz. Comment. Math. Helv. **16**, 264–265 (1944).

Very short presentation of Gauss's fifth proof of the reciprocity law. *A. Brauer* (Chapel Hill, N. C.).

Bose, R. C., Chowla, S., and Rao, C. R. On the integral order, (mod p) of quadratics x^2+ax+b, with applications to the construction of minimum functions for $GF(p^2)$, and to some number theory results. Bull. Calcutta Math. Soc. **36**, 153–174 (1944). [MF 12415]

For p an odd prime, a and b arbitrary integers, the "integral order" of $x^2+ax+b \pmod p$ is defined as the least positive integer such that $x^n \equiv c \pmod{p, x^2+ax+b}$, where c is some integer. The quadratic is not assumed irreducible. Various theorems concerning the integral order are proved. For example, given $b \not\equiv 0$, there are just $\varphi(n)$ irreducible quadratics of integral order n, where $n \mid p+1$ and satisfies certain auxiliary conditions. The values of a satisfy a congruence $f_n(a, b) \equiv 0 \pmod p$, where $f_n(a, b)$ is defined in terms of the cyclotomic polynomial; properties of this congruence are discussed in some detail. Again, given $a \not\equiv 0$, there are exactly $\frac{1}{2}\varphi(n)$ irreducible quadratics of integral order $n > 2$; the values of b satisfy the same congruence.

There are a number of applications of these results. For example, if $p = 12k \pm 1$ and $t^2 \equiv 3 \pmod p$, then $2+t$ and $2-t$ are quadratic residues of p for k even and nonresidues for k odd. *L. Carlitz* (Durham, N. C.).

Bell, E. T. Separable Diophantine equations. Trans. Amer. Math. Soc. **57**, 86–101 (1945). [MF 11911]

An equation of the type

$$\sum_{i=1}^{n} m_i X_i = 0, \qquad n > 2; \ \prod m_i \neq 0, \tag{1}$$

where the m_i are integral constants and the X_i are independent elementary monomials, is called an extended multiplicative equation. (An elementary monomial is an expression of the form $x_1^{a_1} \cdots x_n^{a_n}$, a_i positive integers, $n > 1$, in which at least one exponent a_i is equal to unity.) By considering first the case in which the X_i are of the form $x_{\alpha 1} x_{\alpha 2} \cdots x_{\alpha\mu}$, where the x_{ij} are independent variables, the author shows that the solution of the Diophantine equation (1) is equivalent to that of a finite system of simple linear Diophantine equations. The solution of the latter is known. A system is said to be separable if its solution is equivalent to one or more of the following: (a) the solution of independent extended equations, (b) the solution of simple systems, (c) the solution of systems of linear Diophantine equations, the total number of systems in (a), (b), (c) being finite. From the results concerning equation (1) it follows that separable equations are solvable. The equation

$$\sum_{i=1}^{n} a_i x_i^2 = \sum_{i=1}^{s} b_i u_i v_i, \tag{2}$$

where the a_i and b_i are integral constants and the x_i, u_i, v_i are independent variables, is a separable equation. The same is true when the left member of (2) is replaced by a general quadratic form. Many well-known problems which have been considered in the literature are solved as special cases of these results. [The following typographical errors were noted. Page 90, line 8, replace "second" by "first"; in formulas (7.6) and (7.101), the upper limits of the first and second summations should be $i-1$ and $n-i$, respectively; formula (8.3) should read $\alpha\alpha_i = m_i X'_{\alpha 1} X_{\alpha 2} \cdots X_{\alpha\mu_i}$.]

I. A. Barnett (Cincinnati, Ohio).

Rosenthall, E. On some special Diophantine equations. Bull. Amer. Math. Soc. **50**, 753–758 (1944). [MF 11291]

Let R be an algebraic number field. Then all solutions of $\xi\eta = \zeta\vartheta$ in integers of R are given by $\xi = \kappa\lambda/e$, $\eta = \mu\nu/e$, $\zeta = \kappa\nu/e$, $\vartheta = \lambda\mu/e$, where κ, λ, μ, ν are arbitrary integers of R and e takes only the finite set of rational integral values each equal to the norm of a representative ideal from each class. This result is proved by the author and used for the proof of the following theorem. Let R be a quadratic number field, ξ an integer of R, and $\bar{\xi}$ the conjugate of ξ. Then all the solutions of $\xi\bar{\xi} = z^{2n+1}$ are given by

$$f^{2n+1}\xi = \varphi_1^{2n+1}\varphi_2^{2n}\bar{\varphi}_2 \cdots \varphi_{n+1}^{n+1}\bar{\varphi}_{n+1}^{n},$$

$$f^2 z = \varphi_1\bar{\varphi}_1\varphi_2\bar{\varphi}_2 \cdots \varphi_{n+1}\bar{\varphi}_{n+1},$$

where $f = e_1^{n+1}e_2^n \cdots e_n^2$, and each of $e_1, e_2, \cdots, e_n$ is equal to the norm of a representative ideal from each class.

A. Brauer (Chapel Hill, N. C.).

Vandiver, H. S. On the number of solutions of certain non-homogeneous trinomial equations in a finite field. Proc. Nat. Acad. Sci. U. S. A. **31**, 170–175 (1945). [MF 12503]

The author finds expressions for the number of solutions of the equations $au^e+bv^f+w^g=0$ and $au^e+bv^f+1=0$ in a finite field F, where a, b are in F and $abuvw \neq 0$. The method is that of a previous paper [same Proc. **30**, 362–367 (1944); these Rev. **6**, 117], where the case $e=f$ of the second equation was treated. As an example, the number of distinct integral solutions (x^4, y^3), modulo p (a prime), of the congruence $ax^4+by^3+1 \equiv 0 \pmod p$ is explicitly given in case $p = 4h+1 > 13$ and $abxy \not\equiv 0 \pmod p$. It follows as a corollary that this congruence always has solutions under these conditions. *H. W. Brinkmann* (Swarthmore, Pa.).

Ljunggren, Wilhelm. Über einige Arcustangensgleichungen die auf interessante unbestimmte Gleichungen führen. Ark. Mat. Astr. Fys. **29A**, no. 13, 11 pp. (1943). [MF 12013]

C. Störmer [Skr. Vid.-Selsk. Christiania. I. **1895**, no. 11 (1895)] determined all the solutions of

$$m \arctan(1/x) + n \arctan(1/y) = k\pi/4$$

in integers m, n, x, y, k. The author solves the same problem for the similar equations

$$m \arctan(\sqrt{3}/x) + n \arctan(\sqrt{3}/y) = k\pi/6,$$
$$m \arctan(\sqrt{2}/x) + n \arctan(\sqrt{2}/y) = k\pi/2.$$

The first of them has 14 and the latter 10 nontrivial solutions. *A. Brauer* (Chapel Hill, N. C.).

Hua, L. K., and Shih, W. T. On the lack of an Euclidean algorithm in $R(\sqrt{61})$. Amer. J. Math. **67**, 209–211 (1945). [MF 12427]

Let p be a prime of the form $24n+13$. It was proved by A. Brauer [Amer. J. Math. **62**, 697–712 (1940); these Rev.

2, 146] that the Euclidean algorithm does not exist in $R(\sqrt{p})$ if $p>109$. For $p=13$ and $p=37$ the algorithm exists. Thus the cases $p=61$ and $p=109$ remained undecided. In this paper it is proved that there is no algorithm for $p=61$. The nonexistence of the algorithm for $p=61$ and $p=109$ was also proved by Rédei [Math. Ann. 118, 588–608 (1942); these Rev. 6, 38]. Rédei's proof is completely different from that given in this paper. *A. Brauer.*

Siegel, Carl Ludwig. The trace of totally positive and real algebraic integers. Ann. of Math. (2) 46, 302–312 (1945). [MF 12404]

The author improves the well-known inequality

$$(n^{-1}(x_1+\cdots+x_n))^n \geqq x_1\cdots x_n,$$

which is true for all positive x_i, by proving that

$$(n^{-1}(x_1\cdots x_n))^n \geqq x_1\cdots x_n\cdot Q,$$

where Q is a number that depends on the product

$$\Delta=\prod_{k<m}(x_k-x_m)^2.$$

In fact

$$Q=\prod_{k=1}^{n-1}(1+(n-k)/(\mu+k-1)),$$

where μ is the positive root of the equation

$$(\Delta/n!)\prod_{k=0}^{n-2}((\mu+k)/(n-k))^{n-k-1}=(x_1\cdots x_n)^{n-1}.$$

This inequality, which is a refinement of one obtained by I. Schur [Math. Z. 1, 377–402 (1918)], is obtained by finding the maximum value of Δ as a function of $x_1, \cdots, x_n$ with the conditions that $x_1+\cdots+x_n$ and $x_1\cdots x_n$ are each equal to fixed numbers and $x_i>0$. As in Schur's paper the inequality is used to discuss the sum of the roots x_i of an algebraic equation $x^n+a_1x^{n-1}+\cdots+a_n=0$ with rational integral coefficients a_i in case the roots are positive and distinct. It is shown that there is a number λ_0, approximately equal to 1.733, such that, for any $\lambda<\lambda_0$, $x_1+\cdots+x_n>\lambda n$ if n is large enough. In particular, it is shown that the sum of the roots is greater than $\frac{3}{4}n$ except for the cases $x_1=1$ and $x_1=x_2=\frac{1}{2}(3\pm\sqrt{5})$. By means of this result the author obtains a refinement of Minkowski's inequality for the discriminant d of an algebraic number field K of degree n in the case that K is totally real. Minkowski's inequality is $d^{1/n}>e^2$, if n is sufficiently large, and the author is able to replace the exponent 2 by $1+\log\frac{1}{2}\pi\lambda_0=2.0017889\cdots>2+\frac{1}{5}\frac{1}{6}\frac{1}{8}$. However, the better inequality $d^{1/n}>2\pi e^{\frac{1}{2}}>e^{2.32}$ had already been obtained by Blichfeldt [Monatsh. Math. Phys. 48, 531–533 (1939); these Rev. 1, 68], who also used Schur's method combined with his own theorem on lattice points. *H. W. Brinkmann* (Swarthmore, Pa.).

Siegel, Carl Ludwig. A mean value theorem in geometry of numbers. Ann. of Math. (2) 46, 340–347 (1945). [MF 12406]

Let Ω_1 be the set of n-rowed square matrices with real elements and determinant $+1$. The author starts from Minkowski's fundamental region K of reduced positive quadratic forms. Using K he defines a fundamental region F for Ω_1 with respect to the proper unimodular group T_1 of all matrices of Ω_1 with integral elements. The Ω_1 space is endowed with a measure ω, so that $\int_F d\omega=1$. Let R be a bounded set in the real Euclidean n-space, measurable in the Jordan sense, and let $f(x)$ be a bounded function integrable in R and zero elsewhere. The main purpose of the paper is to prove the theorem

$$\int_F \sum_{g\neq 0} f(Ag)d\omega=\int_R f(x)\{dx\},$$

where g is an integral vector in Euclidean n-space and $A\in\Omega_1$.

The proof consists of an analysis of the integrals

$$\psi(\lambda)=\int_F \lambda^n \sum_{g\neq 0} f(\lambda Ag)d\omega_1, \qquad 0<\lambda\leqq 1,$$

where the integration is performed with respect to a specially defined measure in Ω_1. The author shows that $\psi(\lambda)$ is independent of λ and derives the result from $\psi(1)=\lim_{\lambda\to 0}\psi(\lambda)$.

An immediate deduction from this result is the following theorem of Hlawka [Math. Z. 49, 285–312 (1943); these Rev. 5, 201]. A matrix A exists in Ω_1 so that

$$\sum_{g\neq 0} f(Ag)\leqq\int_R f(x)\{dx\}+\epsilon.$$

Two simple consequences of the analysis are given. One is a proof of Minkowski's assertion that, if B is an n-dimensional star domain of volume less than $\zeta(n)$, then there exists a lattice of determinant 1 such that B does not contain any lattice point other than 0. The other is an evaluation of the volume of that part of Minkowski's fundamental region K of reduced positive quadratic forms for which the determinant is less than or equal to 1. The paper closes by suggesting generalizations of the main result either by imposing fewer restrictions on $f(x)$ or by using other spaces and groups in place of Ω_1 and T_1, respectively.

D. Derry (Saskatoon, Sask.).

Mordell, L. J. Lattice points in the region $|x^3+y^3|\leqq 1$. J. London Math. Soc. 19, 92–99 (1944). [MF 12555]

The author gives a simple proof that every lattice Λ of determinant $(\frac{3}{4}\frac{3}{4})^{1/6}$ contains a point $\neq(0,0)$ of the set $\mathfrak{R}$: $|x^3+y^3|\leqq 1$. It contains only boundary points of this set, if it is either of basis (a, b), (c, d), or of basis (b, a), (d, c), where a, b, c, d are the real numbers satisfying $b<0$,

$$a^3+b^3=c^3+d^3=(a+c)^3+(b+d)^3=(c-a)^3+(d-b)^3=1.$$

If Λ is different from these two "critical lattices," it has (1) a point P in the region $\mathfrak{R}_1$ external to $\mathfrak{R}$ and bounded by the line from (c, d) to $(c-a, d-b)$; (2) a point P_1 in the region $\overline{\mathfrak{R}}_1$ symmetrical to $\mathfrak{R}_1$ in $x+y=0$; (3) a point P_2 in the region $\mathfrak{R}_2$ external to $\mathfrak{R}$ and bounded by the line through (a, b), $(a+c, b+d)$, and the tangent to $x^3+y^3=1$ at (c, d); (4) a point P_3 in the region $\overline{\mathfrak{R}}_2$ symmetrical to $\mathfrak{R}$ in $x+y=0$. The midpoint of one of the line segments joining two of the points $(0, 0)$, P, P_1, P_2, P_3 must then be a point of Λ; a simple discussion shows that then at least one point of Λ different from $(0, 0)$ is an inner point of $\mathfrak{R}$. The result is equivalent to one of the author's for the region $|x^3-xy^2-y^3|\leqq 1$, which he applied in the theory of cubic binary forms [cf. the same J. 17, 107–115 (1942); 18, 201–210, 210–217, 218–221 (1943); Proc. London Math. Soc. (2) 48, 198–228 (1943); these Rev. 4, 131; 6, 37, 38; 5, 172; and, for related work of Davenport, Proc. Cambridge Philos. Soc. 39, 1–21 (1943); same J. 18, 168–176 (1943); these Rev. 4, 212; 5, 254]. *K. Mahler* (Manchester).

Mordell, L. J. On the geometry of numbers in some non-convex regions. Proc. London Math. Soc. (2) 48, 339–390 (1945). [MF 12046]

Let $f(x, y)$ be defined for $x\geqq 0$, $y\geqq 0$ and have the following properties: (1) $f(x, y)$ is continuous and nonnegative;

(2) $f(x, y)=f(y, x)$ and $f(x, y)=xf(1, y/x)$; (3) $f(1, y)$ is steadily increasing for all y, and $f(1+y, 1-y)$ is steadily decreasing for $0\leqq y\leqq 1$ and is not a constant; (4) the region $x\geqq 0$, $y\geqq 0$, $f(x, y)\geqq f(1, 1)$ is convex. Then unique positive numbers a, b, c exist such that $f(a+b, a-b)=f(a, b)=cf(1, 1)$, $a^2+b^2=2$, $a>b$; moreover $a<2c<2$. From the properties of $f(x, y)$, the region $\mathfrak{R}: f(|x|, |y|)\leqq cf(1, 1)$ is symmetrical in the origin and in the four lines $x=0$, $y=0$, $x+y=0$, $x-y=0$; it contains the origin as an inner point, and it is not convex. The author proves the following general theorem, which is the first of its kind.

Let P be any point on the arc C of the first quadrant boundary of $\mathfrak{R}$ between the two points (b, a) and (a, b); there then exist two unique points Q and R on the second quadrant boundary of $\mathfrak{R}$ such that the four points $OPQR$ form the vertices of a parallelogram, $\Pi(P)$ say. If for all P on C this parallelogram is at least of area 2, then every lattice

$$x=\alpha\xi+\beta\eta, \quad y=\gamma\xi+\delta\eta \qquad (\xi, \eta=0, \pm 1, \pm 2, \cdots)$$

of determinant $|\alpha\delta-\beta\gamma|=2$ contains at least one point of $\mathfrak{R}$ different from 0, i.e. a point $(x, y)\neq(0, 0)$ such that $f(|x|, |y|)\leqq cf(1, 1)$. This result is further the best possible one (i.e. the constant $cf(1, 1)$ cannot be replaced by any smaller number) if there exists at least one position of P on C for which the lattice generated by the vertices of $\Pi(P)$ contains no inner points of $\mathfrak{R}$ except O. To prove this theorem, point sets are constructed abutting on $\mathfrak{R}$ and containing at least one point of every lattice of determinant 2 without inner points of $\mathfrak{R}$; the assertion then follows by combining these points.

When applying this theorem, the discussion of the area of $\Pi(P)$ presents difficulties. The author has carried out the calculations in a number of cases of particular interest, and he finds that each of the following regions contains a point different from O of every lattice of determinant 1: (1) the region

$$|x|+|y|+\{(\lambda^2-1)(x^2+y^2)+2|xy|\}^{\frac{1}{2}} \leqq\{\lambda^2+2+(-\lambda^4+12\lambda^2-4)^{\frac{1}{2}}\}^{\frac{1}{2}},$$

where $1\leqq\lambda<\sqrt{2}$; (2) the region

$$\min(|x|+m|y|, |y|+m|x|)\leqq(1+2m-m^2)(5-4m+m^2)^{-\frac{1}{2}},$$

where $1/\sqrt{3}\leqq m\leqq 1$; (3) the region $|x|^p+|y|^p\leqq 2^{1-\frac{1}{2}p}c^p$, where $0.329\cdots\leqq p<1$, and where c has the same meaning as before; (4) the region $|xy|(x^2+y^2)\leqq 2/\sqrt{17}$, or the equivalent region $|x^4-y^4|\leqq 4/\sqrt{17}$; (5) the region $|xy|(|x|+|y|)\leqq 30/(13)^{\frac{1}{2}}$. The results (1)–(4) are the best possible ones, but (5) is not, in the opinion of the reviewer.

Both the methods and the results of this paper go back to an earlier paper [Proc. London Math. Soc. (2) **48**, 198–228 (1943); these Rev. **5**, 172] of the author, where he studied the two regions $|x^3+x^2y-2xy^2-y^3|\leqq 1$ and $|x^3-xy^2-y^3|\leqq 1$. The paper shows that nonconvex lattice point problems form a very promising subject of research in the geometry of numbers. *K. Mahler* (Manchester).

Segre, B. Lattice points in infinite domains and asymmetric Diophantine approximations. Duke Math. J. **12**, 337–365 (1945). [MF 12606]

Let $K_{a,b}$ be the set of points (x, y) for which $-a<xy<b$, $a>0$, $b\geqq 0$. A lattice Λ is defined to be $K_{a,b}$-admissible if it has no point within $K_{a,b}$ except possibly the origin. The area of a fundamental parallelogram of Λ is called the determinant of Λ and is designated by $d(\Lambda)$. Let $\Delta(K_{a,b})$ be the lower bound of $d(\Lambda)$ for all $K_{a,b}$-admissible lattices. The author deduces, by simple geometric methods, that $\Delta(K_{a,b})\geqq(a^2+4ab)^{\frac{1}{2}}$ and that the equality sign occurs if and only if a/b is integral or $b=0$. When the equality sign holds, he constructs all lattices Λ for which $d(\Lambda)=\Delta(K_{a,b})$. The results are applied to derive inequalities for $\Delta(K)$ when K belongs to some general classes of domains for which $K_{a,b}\subseteq K$. A second application is the following approximation theorem. Let τ be any nonnegative number. Then, for any irrational number ξ, infinitely many rationals m/n exist for which

$$-\frac{1}{n^2(1+4\tau)^{\frac{1}{2}}}<\frac{m}{n}-\xi<\frac{\tau}{n^2(1+4\tau)^{\frac{1}{2}}}.$$

For $\tau=1$ this reduces to a classical result of Hurwitz [Math. Ann. **39**, 279–284 (1891)]. When τ is integral the author determines, by continued fraction methods similar to those of Hurwitz, the class of numbers ξ for which this is the best possible approximation. *D. Derry* (Saskatoon, Sask.).

Mahler, K. A theorem of B. Segre. Duke Math. J. **12**, 367–371 (1945). [MF 12607]

The author uses methods developed by Mordell [cf. J. London Math. Soc. **18**, 201–210 (1943); these Rev. **6**, 37] to prove the result of the paper reviewed above that the set of points (x, y) for which $-1\leqq xy<0$ contains a point of the lattice Λ if $d(\Lambda)\leqq 1$. He discusses the connection of this result with continued fractions used in the theory of binary quadratic forms. *D. Derry* (Saskatoon, Sask.).

Whitehead, R. F. A rational parametric solution of the cubic indeterminate equation $z^2=f(x, y)$. J. London Math. Soc. **19**, 68–71 (1944). [MF 12551]

A simple proof is given of the existence of a rational parametric solution of the equation $z^2=ax^3+bx^2y+cxy^2+dy^3+\phi$, where ϕ denotes $ex^2+fxy+gy^2+hx+ky+m$, and all coefficients are rational. The problem reduces to the case

$$z^2=x^3+x(a_1y^2+b_1y+c_1)+d_1y^3+e_1y^2+f_1y+g_1.$$

Putting $x=y^2+2ty$ gives $z^2=(y^3+3ty^2+\cdots)^2+Py^2+Qy+R$, with P, Q, R certain polynomials in t. Solving $Py^2+Qy+R=0$ gives expressions of the form $y=\beta\pm\beta'\sqrt{\Delta}$, $x=\alpha\pm\alpha'\sqrt{\Delta}$, $z=\gamma\pm\gamma'\sqrt{\Delta}$; the straight line $x=\alpha+\alpha'\theta$, $y=\beta+\beta'\theta$, $z=\gamma+\gamma'\theta$ intersects the cubic surface in points given by $\theta=\pm\sqrt{\Delta}$, and in a third point with rational coordinates. Unless all of a_1, b_1, d_1, e_1, f_1 vanish (when $f(x, y)$ reduces to a function of one variable), this point is not at infinity. Segre gave a geometrical proof in the case $\phi=(1+ex+fy)^2$ [same J. **18**, 24–31, 88–100 (1943); these Rev. **5**, 154], Mordell an algebraic proof with $\phi=g(1+ex+fy)^2$ [same J. **18**, 43–46, 222–226 (1943); these Rev. **5**, 154; **6**, 37], and Segre extended Mordell's method to the general case [same J. **18**, 226–233 (1943); these Rev. **6**, 37]. *G. Pall.*

Chowla, S. Contributions to the theory of the construction of balanced incomplete block designs. Math. Student **12**, 82–85 (1945). [MF 12488]

A set of g integers $d_1, d_2, \cdots, d_g$ is called a difference set mod n if the number of solutions of the congruence $m\equiv d_i-d_j(\mathrm{mod}\ n)$ is independent of m provided $m\not\equiv 0(\mathrm{mod}\ n)$. The author states various theorems on difference sets, asserts that he has proved them and verifies them in special cases. *H. B. Mann* (Providence, R. I.).

Chowla, S., and Singh, Daljit. A perfect difference set of order 18. Math. Student 12, 85 (1945). [MF 12489]

The authors give 18 residues $a_i<307$ so that every residue not congruent to 0(mod 307) can be expressed uniquely in the form a_i-a_j, $i\neq j$. *P. Erdös.*

Livingood, John. A partition function with the prime modulus $P>3$. Amer. J. Math. 67, 194–208 (1945). [MF 12426]

Convergent series are found for $p_a(n)$, the number of partitions of n into summands of the form $pl\pm a$, where p is a fixed prime greater than 3. The method used is a direct extension of the work of J. Lehner for $p=5$ [Duke Math. J. 8, 631–655 (1941); these Rev. 3, 166]. For $p<19$ the resulting series are of the same sort as Lehner's, while new terms of a similar type arise for larger p. *H. S. Zuckerman.*

Misra, Rama Dhar. On lattice sums for closest packing crystals. Proc. Benares Math. Soc. (N.S.) 4, 109–112 (1943). [MF 10356]

Let ν_q denote the number of ways of expressing an even number q as the sum of three squares (or as the sum of squares of four numbers whose sum is zero); for example, $\nu_4=6$ from the permutations of $(\pm2)^2+0^2+0^2$ or of $1^2+1^2+(-1)^2+(-1)^2$, but $\nu_{28}=0$. The author tabulates ν_q for $q\leqq50$ in order to investigate the sums

$$S_n=2^{-\frac{1}{2}n}\nu_2+4^{-\frac{1}{2}n}\nu_4+6^{-\frac{1}{2}n}\nu_6+\cdots,$$

which he tabulates for $n=4, 5, \cdots, 15$. He remarks that, for large values of n, the first term alone (namely, $12/2^{\frac{1}{2}n}$) provides a good approximation to S_n.

Geometrically, ν_q is the number of spheres distant $(2q)^{\frac{1}{2}}$ radii from a given sphere (center to center) in any closest packing. [In the first paragraph on page 110, the descriptions of the cubic and hexagonal packings have unhappily been interchanged [cf. W. W. R. Ball, Mathematical Recreations and Essays, 11th ed., Macmillan, New York, 1939, p. 149].] *H. S. M. Coxeter* (Toronto, Ont.).

Rankin, R. A. On the representations of a number as a sum of squares and certain related identities. Proc. Cambridge Philos. Soc. 41, 1–11 (1945). [MF 12370]

Define $s_r=\sum_{i=1}^{4}x_i^r$ and $t_1=s_1$, $t_2=s_2$, $t_3=s_3-\frac{3}{4}s_2s_1+\frac{1}{8}s_1^3$, $t_4=s_4-s_3s_1$. If G is any function of t_1, t_2, t_4, and an odd function of t_3, and if n is any positive even integer, then $\sum G(t_1, t_2, t_3, t_4)=0$, the sum being over all x_1, x_2, x_3, x_4 which are solutions of $\sum_{i=1}^{4}x_i^2=n$. The x_i may be positive, negative, or zero. This is a typical one of several identities proved. Whereas this one involves a Diophantine equation with a sum of four squares, other identities involve eight squares and, more generally, $4c$ squares. Some identities are summations over those particular solutions of the Diophantine equation which satisfy $x_i\equiv1$ (mod 4) for all i. The results are obtained by elementary methods, and it is shown that they may also be derived by equating coefficients in elliptic function expansions. *I. Niven.*

Ward, Morgan. Euler's three biquadrate problem. Proc. Nat. Acad. Sci. U. S. A. 31, 125–127 (1945). [MF 12323]

This famous unsolved problem is to prove or disprove the existence of three positive integral fourth powers whose sum is a fourth power. The author indicates a method by means of which he has shown that the Diophantine equation

$$x^4+y^4+z^4=w^4, \qquad xyz\neq0,$$

has no solution for $w<10000$. This equation is replaced by another equation

$$u^4+v^4=2ekl(e^2l^2+2^{12+6^r+3^s}d^2k^2), \tag{1}$$

whose derivation is not given and which the reviewer fears is misprinted. With $w<10000$, the new variables are subject to inequalities and congruence conditions with respect to the moduli 8, 5 and 13. There results a finite set of numbers represented by the right member of (1). To show that none of these numbers is the sum of two biquadrates each number is expressed in all possible ways as the sum of two squares with the aid of factor tables and the factor stencils of D. N. Lehmer and J. D. Elder. The details of the proof are promised in another paper. *D. H. Lehmer.*

Whiteman, Albert Leon. A note on Kloosterman sums. Bull. Amer. Math. Soc. 51, 373–377 (1945). [MF 12513]

This note is concerned with the important exponential sum

$$A_k(n)=\sum \exp\{2\pi in(h+\bar{h})/k\}, \quad h\bar{h}\equiv1 \pmod{k}, \tag{1}$$

summed over all integers $h\leqq k$ and prime to k. The problem of obtaining a more explicit formula for $A_k(n)$ is not completely solved. The sum is a "multiplicative function" of k, so that we may restrict k to the case of a power of a prime, p^α. For $\alpha=1$ no essentially simpler formula than (1) is known for $A_k(n)$. For $\alpha>1$, Salié [Math. Z. 34, 91–109 (1931)] gave formulas for $A_{p^\alpha}(n)$ expressing it as a single term. The present paper gives a new derivation of these results. The proof is direct rather than by induction over α. The proof fails for $\alpha=1$. The author's method applies also to certain character sums considered by Salié and the reviewer. *D. H. Lehmer* (Aberdeen Proving Ground, Md.).

Pillai, S. S. Bertrand's postulate. Bull. Calcutta Math. Soc. 36, 97–99 (1944). [MF 11841]

Pillai, S. S. Correction to my paper "Bertrand's postulate". Bull. Calcutta Math. Soc. 37, 27 (1945). [MF 13306]

A proof is given of Bertrand's postulate (that if $n\geqq2$ there exists a prime p with $n<p\leqq2n$), based on the ideas of Tschebycheff's original proof. *H. Davenport* (London).

Kostandi, G. Primzahlen der form $kx\pm1$. Bull. Math. Soc. Roumaine Sci. 44, 21–34 (1942). [MF 12747]

An attempt to prove Dirichlet's theorem for the progressions $kx\pm1$ by elementary methods without the use of primitive roots of unity. The proof is not correct.

H. S. Zuckerman (Seattle, Wash.).

Linnik, U. V. On the possibility to avoid Riemann's extended hypothesis in the investigation of primes in progressions. C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 135–138 (1944). [MF 12031]

The author outlines a method of attack on the problem of the magnitude of the least prime in a progression $Dx+l$ in which he assumes, instead of the generalized Riemann hypothesis, a weaker "density property." This assumption is that each of the Dirichlet L-functions (mod D), with

possibly a bounded number of exceptions, has the property that the ratio of the number of zeros in any one circle of radius $1/\log D$ with center in the rectangle $9/10 \leqq \sigma \leqq 1$, $|t| \leqq \log^3 D$ to the number of zeros in any other such circle does not exceed an absolute constant. This paper is superseded by that reviewed in the second review below, where the need for this assumption is surmounted.

H. Davenport (London).

Linnik, U. V. On Dirichlet's *L*-series and prime-number sums. Rec. Math. [Mat. Sbornik] N.S. 15(57), 3–12 (1944). (English. Russian summary) [MF 12279]

In a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 145–146 (1943); these Rev. 6, 58] the author stated an upper bound for the number of zeros of L-functions (mod D) which have a zero in the rectangle $1-\theta \leqq \sigma \leqq 1$, $|t| \leqq \log^3 D$. [In the previous paper the last inequality read $|t| \leqq D^{3/2}$, which was presumably a misprint.] In the present paper he proves a stronger result, which is a step toward a later paper [see the following review]. Let $\lambda > 0$ be arbitrarily small, and let θ satisfy $\frac{1}{2} \geqq \theta \geqq (\log D)^{\lambda-1}$. Let Q denote the number of L-functions (mod D) which have a zero in the rectangle $1-\theta \leqq \sigma \leqq 1$, $|t| \leqq \log^3 D$. Then $Q < \exp(A(\lambda)\theta \log D)$, where $A(\lambda)$ is a positive constant depending only on λ. The proof seems to be very difficult reading.

H. Davenport (London).

Linnik, U. V. On the least prime in an arithmetic progression. I. The basic theorem. Rec. Math. [Mat. Sbornik] N.S. 15(57), 139–178 (1944). (English. Russian summary) [MF 12285]

The author's object is to prove, without any hypothesis, that the least prime in an arithmetic progression $Dx+l$ (where l, D are coprime, and $1 \leqq l \leqq D-1$) does not exceed D^{C_0}, where C_0 is an absolute constant. This first section is devoted mainly to the proof of the "basic theorem." Let $\psi(D)$ be any number satisfying $\frac{1}{2}\log D \geqq \psi(D) \geqq 2$. Let $\theta = \psi(D)/\log D$ and let Q denote the number of L-functions (mod D) which have a zero in the rectangle $1-\theta \leqq \sigma \leqq 1$, $|t| \leqq \min\{\log^3 D, \psi^{100}(D)\}$. Then $Q < \exp(A\psi(D))$, where A is an absolute constant. The proof appears to be very formidable, and is particularly difficult to follow because a detailed knowledge of the author's previous papers is assumed. [Cf. the preceding and following reviews.]

H. Davenport (London).

Linnik, U. V. On the least prime in an arithmetic progression. II. The Deuring-Heilbronn phenomenon. Rec. Math. [Mat. Sbornik] N.S. 15(57), 347–368 (1944). (English. Russian summary) [MF 12302]

[Cf. the preceding review.] The author proves a "second basic theorem" which is a new formulation of a phenomenon discovered by Deuring [Math. Z. 37, 405–415 (1933)] and Heilbronn [Quart. J. Math., Oxford Ser. 5, 150–160 (1934)]. It is well known that there exists a constant c_0 such that there are no zeros of any L-functions (mod D) in the rectangle $1 \geqq \sigma \geqq 1 - c_0/\log D$, $|t| \leqq D$, except possibly one real zero β_0 of a particular L-function formed with a real character. Let $\sigma_0 = 1-\beta_0$ and $\mu_0 = 1/(\sigma_0 \log D) \geqq 1$. Then the rectangle $1 \geqq \sigma \geqq 1 - c_3 \log(c_4\mu_0)/\log D$, $|t| \leqq \log^3 D$ contains no zero of any L-function formed with any modulus not exceeding D^2, except for the zero β_0 of the particular function. Using this and the results of section I, the author proves his main theorem on the least prime in the progression $Dx+l$.

H. Davenport (London).

Linnik, U. V. On the distribution of characters. C. R. (Doklady) Acad. Sci. URSS (N.S.) 42, 323–325 (1944). [MF 11640]

The author announces two results on the distribution of the values of characters (mod D). One is that, if $\chi(n)$ is a complex character (mod D) of bounded order, then

$$N^{-1} \sum_{p>2} \chi(p) \log(p \exp(-p/N)) = O\{(\log D)^{-1} + \exp(-\log N/\log D)\}.$$

The other result is for the character $(-D/n)$, where $-D$ is a fundamental discriminant and differs in having an additional term $\exp(-h \log N/\sqrt{D})$ on the right, where h is the class number of the field $k(\sqrt{(-D)})$. The proof appears to be very deep, and the outline given is too short to be intelligible to the reviewer. In an addendum the author states that he has proved that the least prime equivalent to l (mod D), where $(l, D) = 1$, is less than $\exp(c \log D \log\log\log D)$, where c is a constant.

H. Davenport (London).

Wintner, Aurel. The densities of ideal classes and the existence of unities in algebraic number fields. Amer. J. Math. 67, 235–238 (1945). [MF 12429]

Let $F(n; C)$ denote the number of integral ideals which are contained in an ideal class C of a fixed algebraic number field K and have a norm not exceeding n. It was proved by Dirichlet, for a quadratic field K, and by Dedekind, for an arbitrary K, that the ratios $f(n; C)/n$ $(n = 1, 2, \cdots)$ tend to a finite positive limit $\lambda = \lambda_K$ independent of the choice of C in K; their proofs made use of the existence of a complete system of fundamental unities in K. The author observes that Hecke's functional equation for

$$\zeta(s; C) = \sum_{n=1}^{\infty} \{f(n; C) - f(n-1; C)\} n^{-s}$$

can be established without reference to the theory of unities; since $\zeta(s; C)$ is regular on the line $\sigma = 1$ except for a pole of first order at the point $s = 1$, with a residue λ independent of C, the Dirichlet-Dedekind theorem follows from the results of Wiener and Ikehara concerning Laplace transforms.

C. L. Siegel (Princeton, N. J.).

Wintner, Aurel. Number-theoretical almost-periodicities. Amer. J. Math. 67, 173–193 (1945). [MF 12425]

The author proves, among others, the following results. Let $f(n)$ be a function. Consider the unique $f'(n)$ which satisfies $f(n) = \sum_{d|n} f'(d)$. Assume that $\sum |f'(k)|/k < \infty$. Then $f(n)$ is almost periodic B, and it has the Fourier series $f(n) \sim \sum_{r=1}^{\infty} a_r c_r(n)$, where

$$c_r(n) = \sum_{1 \leqq s \leqq r}^{(s,r)=1} \epsilon(ns/\pi), \qquad \epsilon(x) = e^{2\pi i x},$$

and $a_r = \sum_{r|k} f'(k)/k$. An analogous theorem is proved for the representation $f(n) = \sum f^*(l)$, where the sum is over $1 \leqq l < n$, $(l, n) = 1$, if $\sum_{k=1}^{n} |f^*(k)|\varphi(k)/k$ converges. Another interesting result is that, if $f(n)$ is almost periodic B (or even if $f(n)$ and $|f(n)|$ have mean values), then $\sum f'(k)/k$ converges.

P. Erdős (Stanford University, Calif.).

ANALYSIS

Theory of Functions of Complex Variables

***Bieberbach, Ludwig. Lehrbuch der Funktionentheorie. I. Elemente der Funktionentheorie. II. Moderne Funktionentheorie.** Chelsea Publishing Company, New York, 1945. Band I, xiv+322 pp. $3.25; Band II, vi+370 pp. $3.25.

Photographic reprint of the fourth edition of Band I and the second edition of Band II. The originals were published by Teubner, Leipzig-Berlin, 1934 and 1931.

Torroja Caballé, Eduardo. Geometric representation of imaginary elements. Revista Mat. Hisp.-Amer. (4) **3**, 338–350 (1943). (Spanish) [MF 12175]

An expository lecture.

Sanvisens, Francisco. The indicatrices of functionals of analytic n-vectors and their application to the integration of rational functions. Revista Mat. Hisp.-Amer. (4) **4**, 60–70 (1944). (Spanish) [MF 12164]

The author considers problems in the theory of linear functionals defined over a space of elements $\{y_1, y_2, \cdots, y_n\}$, where $y_i = y_i(t)$ is a function analytic over some region of the complex sphere. He studies relations between the various indicatrices of such functionals and makes applications to the integration of rational functions. The results revolve about the Cauchy formula. *E. R. Lorch.*

Markouchevitch, A. Prolongement analytique et ultraconvergence. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 223–225 (1944). [MF 12663]

Using his earlier results [same C. R. (N.S.) **41**, 227–229 (1943); **44**, 262–264 (1944); Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 433–436 (1944); these Rev. **6**, 69, 179, 274], the author shows that there exist a sequence of real numbers λ_j, with $\limsup \lambda_j^{1/j} = \limsup |1-\lambda_j|^{1/j} = 1$, and increasing sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that, for every $f(z) = \sum a_n z^n$ analytic at $z=0$,

$$f(z) = \lim_{k\to\infty} \sum_{j=0}^{n_k} \lambda_j a_j z^j + \lim_{k\to\infty} \sum_{j=0}^{m_k} (1-\lambda_j) a_j z^j,$$

uniformly in every closed subset of the Mittag-Leffler star of $f(z)$. If $f(z)$ is analytic in a simply-connected region D containing $z=0$, then either $\sum a_n z^n$ is overconvergent in D or there exists a sequence $\{\lambda_j\}$, depending on $f(z)$ and on D, such that both series $\sum \lambda_j a_j z^j$ and $\sum (1-\lambda_j) a_j z^j$ are overconvergent in D. The second result can be generalized to developments in terms of functions which form a base (in the sense of Schauder) for functions analytic in a subregion of D. *R. P. Boas, Jr.* (Providence, R. I.).

Vignaux, J. C., and Cotlar, Mischa. Totally normal families of holomorphic functions. Publ. Inst. Mat. Univ. Nac. Litoral **7**, 152 pp. (1944). (Spanish. French summary) [MF 12208]

Whereas the Montel theory of normal families $\{f(z)\}$ of analytic functions of a complex variable is concerned with properties and criteria relative to regions completely interior to the domain D of definition, the present development is concerned also with convergence of subsequences on the boundary or part of the boundary of D. The family $\{f(z)\}$ of functions analytic in D: $|z|<1$ and possessing radial limit values at points of a subset E of $|z|=1$ is said to be totally normal in $D+E$ if it is normal in the sense of Montel in D and if, in addition, in each sequence in $\{f(z)\}$ there exists a subsequence for which the boundary values converge on suitably defined subsets of E. Since this latter convergence might be prescribed to be simple, uniform, weak, in the mean, and so on, several sorts of total normality can be considered; in all, seven are presented. The authors give first a preparatory discussion of families of real functions and the Fréchet theory of compact sets in function-space, followed by various extensions of theorems of Montel, Ostrowski, Khintchine and others, before developing criteria and properties of totally normal families of analytic functions of a complex variable.

E. F. Beckenbach (Los Angeles, Calif.).

***Littlewood, J. E. Lectures on the Theory of Functions.** Oxford University Press, 1944. 243 pp. $5.50.

This book is devoted to a group of topics in function theory which center around the problem of conformal mapping. A certain amount of familiarity with the basic concepts of both real and complex variables is assumed. The material is divided into three parts: the introduction, which develops the necessary real variable theory; chapter I, which contains classical material on the maximum modulus principle and culminates in a proof of Riemann's mapping theorem; and chapter II, in which specialized topics are treated, such as functions which omit certain values, and functions schlicht in the unit circle. The first two parts were printed in 1931, but additions and corrections have been made in the present volume. There is no index. We proceed to a more detailed account of the contents.

Introduction: inequalities of the Hölder and Minkowski type, including the Riesz convexity theorem; properties of the Lebesgue integral, such as the Egoroff theorem, absolute continuity, and types of convergence; a brief account of Fourier series; a fairly complete treatment of the basic properties of harmonic functions of two variables, including theorems of Fatou type.

Chapter I: the maximum modulus principle; classical theorems like Schwarz's lemma and Hadamard's three-circles theorem; the theory of the linear function $w=(az+b)/(cz+d)$; a proof of the fundamental theorem on the conformal mapping of simply connected regions; behavior of mapping functions on the boundary.

Chapter II: subharmonic functions, with application to the mean value of functions analytic in a circle; a thorough discussion of the Lindelöf principle of subordination, with applications to theorems of Picard type; the classical theorems on schlicht functions, stopping short of Loewner's theorem on the third coefficient. *H. Pollard.*

Santaló, Luis A. A theorem on conformal mapping. Math. Notae **5**, 29–40 (1945). (Spanish) [MF 12618]

Let

$$z' = f(z) \tag{1}$$

be an analytic function mapping a domain D conformally on D', P a point of D, γ a smooth curve through P, $\zeta = \xi + i\eta$ the center of curvature of γ at P, and P' and $\zeta' = \xi' + i\eta'$ the images of P and ζ, respectively, under the transformation (1). A computation shows that the equations for ξ' and η' in terms of ξ and η have coefficients depending only on P and $f(z)$; thus ζ' is independent of the

particular curve γ through P with ζ as center of curvature at P. Furthermore, the equations giving ξ' and η' in terms of ξ and η are those of a projective transformation. Thus the point P and the function (1) determine a projective transformation of the entire projective plane of the domain D on that of D', with correspondence of ζ and ζ'.

Several implications of the result are pointed out. If three curves are tangent at P, and K_1, K_2, K_3 are their curvatures at P, then the expression $(K_1-K_3)/(K_1-K_2)$ is invariant under the transformation (1). Again, the image of the family of straight lines through P under the transformation (1) is a family of curves such that their centers of curvature at P' all lie on a straight line.

E. F. Beckenbach (Los Angeles, Calif.).

Goodman, Ruth E. On the Bloch-Landau constant for schlicht functions. Bull. Amer. Math. Soc. **51**, 234–239 (1945). [MF 12259]

In this note an upper bound is given for the Bloch-Landau constant $\mathfrak{A}$. The best previous result is due to R. M. Robinson [same Bull. **41**, 535–540 (1935)], who showed that $\mathfrak{A}<0.658$. The procedure employed by Robinson is extended in the present note and it is shown that $\mathfrak{A}<0.65647$. Properties of the mapping function employed in obtaining this estimate are studied. *M. H. Heins.*

Denjoy, Arnaud. Sur la représentation conforme des aires planes. Mathematica, Timişoara **20**, 73–89 (1944). [MF 12452]

A number of results in the theory of conformal mapping are obtained by the author with the aid of the following lemma. If c denotes a closed circular disc with circumference γ and if $f(z)$ denotes a function which is analytic on c, then

$$\int_\gamma |dz_2| \int_\gamma |(f(z_2)-f(z_1))/(z_2-z_1)|^2 |dz_1| = 4\pi \int\int_c |f'(z)|^2 dS.$$

The major theorems of the paper are as follows. (I) Let $f(z)$ be analytic and bounded for $|z|<1$ and such that the area of the Riemannian image of $|z|<1$ with respect to $f(z)$ is finite. Further, let E denote the set of measure 2π of points ζ of $|z|=1$ at which f possesses an angular limit (denoted by $f(\zeta)$). Then there exists a subset H of E which is also of measure 2π and is such that, for every point ξ of H,

$$\int_E |(f(\zeta)-f(\xi))/(\zeta-\xi)|^2 |d\zeta| < \infty.$$

(II) Let $f(z)$ be analytic and univalent in $|z|<1$, and at each point ζ of $|z|=1$ where $f(z)$ possesses a finite angular limit let this limit be denoted by $f(\zeta)$. Then the set E of points ζ is a set of measure 2π and there exists a subset H of E, also of measure 2π, with the following property: if ξ is in H and ϵ and A are two arbitrary positive numbers, then (1) the set $I(\xi,\epsilon)$ of points ζ satisfying $|f(\zeta)-f(\xi)|>\epsilon$ has at the point ξ a density of order one which is null, so that

$$\int_{I(\xi,\epsilon)} |d\zeta|/|\zeta-\xi|^2 < \infty,$$

and (2) on $E-I(\xi, A)$ the difference quotient

$$(f(\zeta)-f(\xi))/(\zeta-\xi),$$

considered as a function of ζ, possesses an integrable square. The relation of the results of this paper to the anterior research of the author, J. Wolff and J. Ferrand is indicated.

M. H. Heins (Arlington, Va.).

Ferrand, Jacqueline. Sur les fonctions holomorphes dans une couronne. Bull. Sci. Math. (2) **67**, 42–49 (1943). [MF 12629]

A simple proof is given of the following theorem of Golusin [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 617–619 (1937)]. Let $f(z)$ be analytic and univalent for $|z|<1$ and satisfy $f(0)=0$, $f'(0)=1$. If $\rho(\theta)$ denotes the distance from the origin of the w-plane to the nearest point of the ray $\arg w=\theta$ which intersects the boundary of the image of $|z|<1$ with respect to $f(z)$, then

$$\int_0^{2\pi} \log\rho \, d\theta \geqq 0.$$

The proof is based on the methods employed by Ahlfors in his thesis and employs Schwarz's inequality applied to appraisals of length and area. The theorem is extended to the case where $f(z)$ is analytic but not necessarily univalent for $|z|<1$. *M. H. Heins* (Arlington, Va.).

Heins, Maurice H. The problem of Milloux for functions analytic throughout the interior of the unit circle. Amer. J. Math. **67**, 212–234 (1945). [MF 12428]

The problem is as follows. Let E be a point set of the z-plane, lying in and closed relative to $|z|<1$, having the property that, for all R satisfying $0\leqq R<1$, the intersection of E with $|z|=R$ is not void. A function $f(z)$ will be said to belong to the class $\mathfrak{F}_m$ if $f(z)$ is defined, analytic, and of modulus less than one for $|z|<1$ and if there exists an E such that $z\in E$ implies $|f(z)|\leqq m$, where $0<m<1$. Let $M(f;r)$ denote $\max_{|z|=r}|f(z)|$. It is required to determine l.u.b. $M(f;r)$ for $f\in\mathfrak{F}_m$ and the associated extremal functions. This problem is related to the following result of Milloux. If $f(z)$ is analytic and $|f(z)|\leqq 1$ in $|z|\leqq 1$ and if there exists a Jordan arc J in $|z|<1$, connecting $z=0$ with a point of $|z|=1$, such that $|f(z)|\leqq m<1$ for $z\in J$, then k, independent of m and J, exists such that $|f(z)|<m^{k(1-|z|)}$ for $|z|<1$.

The problem is first reduced by normalizing the extremal function $f_0(z)$ (which obviously exists) so that $M(f_0;r)$ is assumed on the positive real axis. With this normalization, it is proved that the portion of the negative real axis in $|z|<1$ is an associated E-set to f_0. The extremal functions corresponding to this E-set have been studied qualitatively by the author [Trans. Amer. Math. Soc. **55**, 349–372 (1944); these Rev. **5**, 259] and a further investigation shows that the extremal functions are independent of r. Two methods are given for determining the extremal functions. The first is a successive composition of rational functions, which leads to a sequence of rational functions converging to the extremal function. The second involves the use of Blaschke products which lead to an expression of the extremal functions in terms of Jacobi theta functions; this requires solving for a parameter defined implicitly by an equation involving theta functions. The methods developed are applied to a number of other extremal problems and to hyperbolic analogues of the Chebyshev polynomials.

J. W. Green (Los Angeles, Calif.).

Frostman, Otto. Sur les produits de Blaschke. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **12**, no. 15, 169–182 (1942). [MF 12589]

This paper is concerned with the existence of radial limits of modulus one and the existence of angular derivatives for convergent Blaschke products. Let $\{a_k\}$ $(k=1, 2, \cdots)$ de-

note a sequence of complex numbers of modulus less than one and let d_k denote $1-|a_k|$ $(k=1, 2, \cdots)$. It is assumed that $\sum_{k=1}^{\infty} d_k < +\infty$. The Blaschke product $B(z)$ defined by

$$B(z)=\prod_{k=1}^{\infty}\frac{\bar{a}_k}{|a_k|}\frac{a_k-z}{1-\bar{a}_k z}$$

is then convergent for $|z|<1$. The following is typical of the results established. A necessary and sufficient condition that the Blaschke product $B(z)$, as well as all Blaschke products having subsets of the $\{a_k\}$ as their zeros, possess radial limits of modulus one at the point $e^{i\theta}$ is that

$$\sum_{k=1}^{\infty} d_k/|e^{i\theta}-a_k| < +\infty.$$

The proof is based on a series of elementary inequalities concerning the individual factors of the product. Related global theorems are obtained from this local theorem by introducing the concept of capacity of order α, α being a real positive number less than one. A cycle of analogous theorems is obtained for the angular derivative of a Blaschke product. An example is given of a bounded function whose derivative is not "beschränktartig." The question of the existence of such bounded functions was raised by Bloch.

M. H. Heins (Arlington, Va.).

Walker, A. G. Note on integral functions. J. London Math. Soc. **19**, 106–107 (1944). [MF 12557]

By an elementary proof the author shows that every real number is a zero of an integral function $f(x)$ whose power series $\sum_0^{\infty} a_n x^n$ has rational coefficients. *M. S. Robertson.*

Shah, S. M. The lower order of the zeros of an integral function. II. Proc. Indian Acad. Sci. Sect. A. **21**, 162–164 (1945). [MF 12645]

In part I [J. Indian Math. Soc. (N.S.) **6**, 63–68 (1942); these Rev. **4**, 137] the author gave an inequality connecting the lower order (λ) and the lower order of the zeros (λ_1) of a function of order ρ, $0<\rho<1$. He now proves that, if ρ is not an integer,

$$\lambda \leqq p+\frac{(\rho-p)\lambda_1}{\lambda_1(\rho-p)+\rho(p+1-\rho)},$$

where p is the genus. *R. P. Boas, Jr.*

Shah, S. M. A note on meromorphic functions. Math. Student **12**, 67–70 (1945). [MF 12483]

The principal object of this paper is the following theorem. Let $f(z)$ be a meromorphic function of order $\rho<1$, and let a and b be distinct complex numbers, one of which may be ∞; then, in the customary Nevanlinna notation,

$$\liminf_{r\to\infty}\frac{T(r)}{N(r, a)+N(r, b)} \leqq 1/(1-\rho).$$

O. Helmer (New York, N. Y.).

Levinson, Norman. Corrections to "The Gontcharoff polynomials." Duke Math. J. **12**, 335 (1945). [MF 12605]

The paper in question appeared in the same J. **11**, 729–733 (1944); cf. these Rev. **6**, 122.

Hughes, H. K. The asymptotic developments of a class of entire functions. Bull. Amer. Math. Soc. **51**, 456–461 (1945). [MF 12528]

The general result of an earlier paper [same Bull. **50**, 425–430 (1944); these Rev. **5**, 234] is applied to obtain an asymptotic expansion at infinity for the function

$$f(z)=\sum_{n=0}^{\infty} z^n/\{\Gamma(\alpha_1 n+s_1)\cdots\Gamma(\alpha_n n+s_m)\}, \quad m \geqq 2,$$

where the constants α_j are real and positive, and the constants s_j may be complex. *I. M. Sheffer.*

Wirtinger, Wilhelm. Integrale dritter Gattung und linear polymorphe Funktionen. Monatsh. Math. Phys. **51**, 101–114 (1944). [MF 12479]

Let G be the Riemann surface of an algebraic function of one variable. A polymorphic function z on G is termed a λ-function if the neighborhoods on G are mapped by z onto schlicht domains. The author investigates the connection of the λ-functions with the elementary integrals of the third kind on G. Let w be an algebraic function or Abelian integral on G. Then the integrand $Q(z_1, z_2)dz_1dz_2$ of an elementary integral of the third kind has the expansion

$$dw_1dw_2[(w_1-w_2)^{-2}+\phi(z)+R_1(w_1-w, w_2-w)],$$

where $\phi(z)$ is an algebraic function of z. Using a modified form,

$$[f]_z=\tfrac{1}{6}(f'''(z)/f'(z))-\tfrac{1}{4}(f''(z)/f'(z))^2,$$

of Schwarz's differential operator, it is proved that $[z]_w=\phi$ is a third-order differential equation of a λ-function z. Conversely, each λ-function z determines a class of integrals of the third kind (the members of the class differ by certain symmetric bilinear forms in the differentials of the first kind dv_α) such that z is a solution of the corresponding differential equation. The differential equation for z is rewritten invariantly in terms of an odd ϑ-function $\vartheta(v)$ of G with $\partial\vartheta/\partial v_\alpha \neq 0$ at the origin $v=0$, and

$$[z]_w=-\tfrac{1}{3}\left[\frac{\partial^3\vartheta(v)}{\partial v_\alpha\partial v_\beta\partial v_\gamma}\right]_{v=0}\frac{dv_\alpha}{dw}\frac{dv_\beta}{dw}\frac{dv_\gamma}{dw}.$$

O. F. G. Schilling (Chicago, Ill.).

Delange, Hubert. Sur le domaine de convergence absolue des séries multiples de puissances. Bull. Sci. Math. (2) **67**, 115–136 (1943). [MF 12635]

The author considers a double power series (*) $\sum a_{mn}w^m z^n$, the summation extending over an infinity of pairs of indices m and n corresponding to all those coefficients a_{mn} which are different from zero. He interprets the domain of absolute convergence of the series as a set S in an (X, Y)-plane so that a point (w, z) $(wz \neq 0)$ is a point of absolute convergence of (*) if and only if the point (ρ, ρ') belongs to the set S, where $\rho=\log|w|$, $\rho'=\log|z|$. He gives various geometric interpretations of the set S and of its boundary. A typical result is obtained in the following manner. To each coefficient $a_{mn}(\neq 0)$ the author associates the line $\Delta_{m,n}: mX+nY+\log|a_{mn}|=0$. He then defines, in a natural manner, the notions of limiting lines of a set of lines and of the positive and negative side of a line. With these two concepts he considers the set of limiting lines Δ of the set $\{\Delta_{mn}\}$ and shows that the series (*) is absolutely convergent in the domain formed by the points (X, Y) situated on the negative side of all the lines Δ, and divergent in the exterior of this domain. Various other geometric interpretations of the domain of absolute convergence are given. The paper closes with similar interpretations for triple series.

W. T. Martin (Syracuse, N. Y.).

Balseiro, José A. Explanation of an omission. Revista Union Mat. Argentina **10**, 173–174 (1945). (Spanish) [MF 12873]

The generalized analytic functions of the author [Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 180, Vol. 3, num. 4. Serie segunda, 14, Contribuciones, 413–442 (1944); these Rev. **6**, 209] are equivalent to those discussed by A. Valeiras [Publ. Circulo Mat. Inst. Nac. Profesorado Secund. no. 5, 1–56 (1939); these Rev. **4**, 140].

Fourier Series and Generalizations, Integral Transforms

Zee-Kiang, Süe-yung. On the variation of increasing functions whose first $2n$ Fourier coefficients are given. J. London Math. Soc. **19**, 71–77 (1944). [MF 12552]

Let $f(\theta)$ be a bounded function increasing in the interval $0 \leqq \theta < 2\pi$, and let $S = f(2\pi - 0) - f(+0) > 0$. Let

$$f(\theta + 2\pi) = f(\theta)$$

and

$$f(\theta) \sim \sum (a_n \cos n\theta + b_n \sin n\theta).$$

Suppose that the function $f(\theta)$ is allowed to change, subject only to the condition that it remains increasing in $0 \leqq \theta < 2\pi$ with finite variation S, and periodic with period 2π. It is shown that the region of variability of the point in $2n$-dimensional space whose coordinates are

$$x_k = 1 + \pi k b_k / S, \quad y_k = -\pi k a_k / S, \qquad 1 \leqq k \leqq n,$$

is (except for a certain "cone" of points) the nth Carathéodory domain, that is, the convex cover K_n of the curve

$$\cos k\theta, \quad \sin k\theta, \qquad 1 \leqq k \leqq n,$$

where θ is a parameter. It follows that, if we keep the first $2n$ Fourier coefficients a_k, b_k $(1 \leqq k \leqq n)$ fixed, then the variation S has a positive minimum S_n^*. It is shown that S_n^* is attained for a function of the set, the remaining Fourier coefficients being determined uniquely by the first $2n$, and that S_n^* can be expressed rationally in terms of the first $2n$ coefficients. *L. S. Bosanquet* (London).

Ríos, Sixto. Note on the convergence of trigonometric series. Mat. Elemental. Madrid (4) **3**, 5 pp. (1943). (Spanish) [MF 12790]

Applying a geometric method suggested by Bouligand, the author proves the well-known result that $\sum a_n \cos nx$ converges if $a_n \to 0$ and $\sum |a_n - a_{n-1}|$ converges.

R. P. Boas, Jr. (Providence, R. I.).

Boas, Ralph P., Jr. Concerning the sequence $\{\cos n_k x\}$, $n_k \to \infty$. Math. Notae **5**, 41 (1945). (Spanish title) [MF 12619]

Letter to the editor, commenting on a paper of Cotlar and Levi [Math. Notae **4**, 145–155 (1944); these Rev. **6**, 120].

Menchoff, D. Sur les sommes partielles des séries de Fourier des fonctions continues. Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 385–432 (1944). (French. Russian summary) [MF 12304]

Complete proofs of results announced previously [C. R. (Doklady) Acad. Sci. URSS (N.S.) **41**, 51–53 (1943); these Rev. **6**, 47]. *A. Zygmund* (Philadelphia, Pa.).

Chen, Kien-Kwong. Criteria for the absolute convergence of a Fourier series at a given point. Amer. J. Math. **67**, 285–299 (1945). [MF 12434]

A typical result is as follows: if $0 < p < 1$ and if the derivative of

$$\int_0^t u^p (t-u)^{-p} \varphi(u)\,du$$

$$\int_0^t u^p (t-u)^{-p} \varphi(u)\,du, \quad 2\varphi(u) = f(x+u) + f(x-u),$$

exists and is of bounded variation in the interval $0 \leqq t \leqq \pi$, the Fourier series of f converges absolutely at the point x. *A. Zygmund* (Philadelphia, Pa.).

Chen, Kien-Kwong. Some relations between the behavior of a function and the absolute summability of its Fourier series. Amer. J. Math. **67**, 239–248 (1945). [MF 12430]

Chen, Kien-Kwong. A generalization of Hardy's theorem with an application to the absolute summability of Fourier series. Amer. J. Math. **67**, 249–255 (1945). [MF 12431]

If the Fourier series of the function $f(t)$ converges absolutely at the point $t = x$, then the function

$$\Phi(t) = t^{-1}\rho(t) \int_0^t \{f(x+u) + f(x-u)\}\,du, \quad 0 \leqq t \leqq A,$$

is of bounded variation in the interval $0 \leqq t \leqq A$, provided that $\rho(t)$ is absolutely continuous and such that

$$\int_0^A t^{-1} |\rho(t)|\,dt < \infty$$

(for example, $\rho(t) = 1/\log^2 t$, $0 < A < 1$). Extensions are given to the case when absolute convergence is replaced by summability.

In the second paper the author studies the inverse problem: the influence of the properties of f on the summability $|C, \alpha|$ of the Fourier series of f. *A. Zygmund.*

Cheng, Min-Teh. The super-absolute Cesàro summability of Fourier series. Duke Math. J. **12**, 409–417 (1945). [MF 12612]

Let σ_r^α be the (C, α) means of the series $\sum u_r$. If $\sum |\sigma_{l_r}^\alpha - \sigma_{l_{r-1}}^\alpha| l_r$ converges, where $l_1 < l_2 < \cdots$, $l_r \to +\infty$, the series $\sum u_r$ is called by the author super-absolutely summable (C, α) with respect to the sequence $\{l_r\}$. This notion is applied to Fourier series. The results are too technical to be reproduced here. *A. Zygmund* (Philadelphia, Pa.).

Lozinski, S. On convergence and summability of Fourier series and interpolation processes. Rec. Math. [Mat. Sbornik] N.S. **14(56)**, 175–268 (1944). (English. Russian summary) [MF 12307]

The paper consists of three chapters, in each of which different problems are treated. In chapter I, § 1, the author investigates the properties of "Fejérian" methods of summability, that is of "triangular" methods summing uniformly the Fourier series of every continuous function $f(x)$ to sum $f(x)$. In §2, properties of Orlicz's spaces generalizing L^p spaces are reviewed [see Orlicz, Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. **8–9**, 207–220 (1932); Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 95]. Let $\Phi(u)$ and $\Psi(u)$ be a pair of functions complementary in the sense of Young, so that $ab \leqq \Phi(a) + \Psi(b)$ for all nonnegative a and b. The space L_Φ^* is defined as the set of all functions $x(t)$, $0 \leqq t \leqq 1$, such that (*) $x(t)y(t)$ is integrable for every $y(t)$ with (**) $\int_0^1 \Psi(|y|)\,dt \leqq 1$. The norm $\|x\|_\Phi$ is defined as sup $\int_0^1 xy\,dt$

for all y's satisfying (*). The space L_ϕ^* then becomes a complete linear metric space. These notions find application in § 3, where the author discusses generalizations of the well-known result of M. Riesz asserting that, if a function f belongs to L^p, $p>1$, so does the conjugate function $\bar{f}$ [see also Trigonometrical Series, p. 165, ex. 6]. Applications to the mean convergence of Fourier series are given. Chapter II is devoted to the theory of interpolation; the author stresses the formal analogy which exists between Fourier series and interpolating polynomials with equidistant fundamental points. With any L-integrable function $f(x)$ of period 2π we may associate two kinds of interpolating polynomials. Either we develop the integral $F(x)=\int f dx$ (which is assumed periodic) into interpolating polynomials and consider the first derivatives of the latter, or we interpolate the mean values of the function over the intervals between consecutive fundamental points. Both approaches have already been tried with the L^p metric; the author now applies Orlicz's metric. Chapter III contains proofs of certain negative results. [The proof given by the reviewer in Trigonometrical Series, p. 96, of the fact that $\|x\|_\phi<\infty$ for all $x\epsilon L_\phi^*$ is declared insufficient by the author. There is, however, no difficulty there. For, we may first define L_ϕ^* as the set of x with $\|x\|_\phi<\infty$, and then, as the proof on p. 99 shows, condition (*) assures the finiteness of $\|x\|_\phi$.] *A. Zygmund.*

Prasad, B. N. The summability of a Fourier series and its conjugate series. Science and Culture **10**, no. 9, Supplement 1 (1945). [MF 12394]

Historical remarks.

Zygmund, A. On the degree of approximation of functions by Fejér means. Bull. Amer. Math. Soc. **51**, 274–278 (1945). [MF 12267]

It is well known that, if the periodic function $f(x)$ belongs to the class Lip α, $0<\alpha<1$, the $(C, 1)$ means $\sigma_n(x)$ of the Fourier series of $f(x)$ satisfy the condition $\sigma_n(x)-f(x)=O(n^{-\alpha})$ uniformly in x. If, however, $\alpha=1$, we can only assert that $\sigma_n(x)-f(x)=O(n^{-1}\log n)$. The main result of the present paper is that, if $f(x)$ is of power series type, $f(x)\sim\sum_0^\infty c_n e^{inx}$, the logarithm can be removed from the last inequality; more generally, if $f(x)\sim\sum_0^\infty c_n e^{inx}$ has the modulus of continuity $\omega(\delta)$, it is proved that $|\sigma_n(x)-f(x)|\leqq A\omega(2\pi/n)$, where A is an absolute constant. The result is also extended to the L^p metric. *R. Salem.*

Linés Escardó, E. On the succession of values of a periodic or almost periodic function at points whose abscissas are in arithmetic progression. Revista Mat. Hisp.-Amer. (4) **4**, 51–59 (1944). (Spanish) [MF 12163]

The author proves the following theorem and generalizations applying to functions of n variables and to Bohr almost periodic functions. Let $f(x)$ be Riemann integrable over $0\leqq x\leqq 1$, let $f(x)$ have period 1, and let a and b be real numbers with a irrational. Then

$$\lim_{n\to\infty}(n+1)^{-1}\sum_{k=0}^{n} f(ka+b)=\int_0^1 f(x)dx.$$

R. P. Agnew (Ithaca, N. Y.).

Kovanko, A. S. Sur la compacité des systèmes de fonctions presque périodiques généralisées de H. Weyl. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 275–276 (1944). [MF 11618]

The author gives necessary and sufficient conditions for the conditional compactness of two types of systems of Weyl almost periodic functions. He also gives, for these functions, generalizations of Bochner's theorem that the class of Bohr almost periodic functions is identical with the class of Bochner's normal functions. *R. H. Cameron.*

Kovanko, A. S. Sur la compacité des systèmes de fonctions presque-périodiques généralisées de A. Besicovitch. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 49–50 (1944). [MF 11608]

The author states necessary and sufficient conditions for the compactness of systems of Besicovitch almost periodic functions. *R. H. Cameron* (Minneapolis, Minn.).

Doss, Raouf. Contribution to the theory of almost periodic functions. Ann. of Math. (2) **46**, 196–219 (1945). [MF 12398]

The author gives necessary and sufficient conditions for the series $\sum_{n=1}^\infty a(u_n)\exp(iu_n x)$ to be the Fourier series of an almost periodic (a.p.) function of a given one of the following types: Bohr a.p. (B_0), Stepanoff a.p. (S), Stepanoff a.p. and bounded (S_b), Weyl a.p. and bounded (W_b), Besicovitch a.p. and bounded (B_b). The conditions are placed on the Bochner sums $\sigma_m(x)$ of the series $\sum a(u_n)\exp(iu_n x)$ and (to quote the two extremes) the condition for (B_0) is that $\sigma_m(x)$ is Bohr homogeneous; for (B_b), that $\sigma_m(x)$ is uniformly bounded.

The author also shows that a necessary and sufficient condition that $\lambda(u)$ is a "multiplicator" (B_0, B_0), (S_b, S_b), (W_b, W_b) or (B_b, B_b) is that $\lambda(u)$ is the Fourier-Stieltjes transform of some function of bounded variation. The real or complex function $\lambda(u)$ is called a multiplicator (B_0, B_0) if, corresponding to any function $f(x)\epsilon(B_0)$ having the Fourier series $\sum a(u_n)\exp(iu_n x)$, there exists a function $F(x)\epsilon(B_0)$ whose Fourier series is $\sum a(u_n)\lambda(u_n)\exp(iu_n x)$; (S_b, S_b), etc., are similarly defined. Related results are also obtained, such as (for instance) that the above condition is sufficient for (S, S) and (W, W). *R. H. Cameron.*

Cossar, J. The Cesàro summability of Fourier integrals. Proc. Edinburgh Math. Soc. (2) **7**, 84–92 (1945). [MF 12352]

If $f(t)$ is integrable in every finite interval, and

$$g(u)=(2/\pi)\int_0^\infty f(t)\sin utdt,$$

the integral being boundedly summable (C, r) in every finite interval, where $r\geqq 0$, then

$$f(x)=\int_0^\infty g(u)\sin xudu$$

for almost all x, the integral being summable $(C, r+1)$. Two more general results are also given, overlapping results of J. L. B. Cooper [Proc. London Math. Soc. (2) **48**, 292–309 (1944); these Rev. **6**, 126]. *R. P. Boas, Jr.*

Boas, R. P., Jr., and Kac, M. Inequalities for Fourier transforms of positive functions. Duke Math. J. **12**, 189–206 (1945). [MF 12079]

The authors establish a number of inequalities for the Fourier transform

$$f(x)=\int_{-\infty}^\infty e^{ixt}\phi(t)dt$$

of a nonnegative function $\phi(t)$, which are analogous to and include some inequalities for the coefficients of positive

trigonometric polynomials established by Fejér and others [cf. Egerváry and Szász, Math. Z. 27, 641–652 (1928)]. In particular, they show that, if $f(x)$ vanishes for $|x| \geqq 1$, then

$$|f(1/n)| \leqq f(0) \cos \pi/(n+1), \quad n=2, 3, \cdots,$$

and, more generally, that the same conclusion holds if

$$|f(kA/n)| = 0, \qquad k=n, n+1, \cdots.$$

These results are also shown to be best possible. Further developments lead to interesting results on the zeros of Fourier transforms. [On p. 197, second line from below, cos should be sin; in (1.4) and (3.4), $[x]$ should mean the least integer not less than x.] *A. C. Offord.*

Duffin, R. J. Representation of Fourier integrals as sums. I. Bull. Amer. Math. Soc. **51**, 447–455 (1945). [MF 12527]

Let $\varphi(x)$ be an arbitrary function, define $F(x(\pi/2)^{\frac{1}{2}})$ and $G(x(\pi/2)^{\frac{1}{2}})$ by the series

$$\sum_{n=0}^{\infty} (-1)^n \varphi(x/(2n+1))/(2n+1),$$

$$\sum_{n=0}^{\infty} (-1)^n \varphi((2n+1)/x)/x,$$

respectively. Then under suitable conditions and for appropriate definition of the sine transform it is true that

$$G(x) = (\pi/2)^{\frac{1}{2}} \int^{\infty} \sin xt F(t) dt.$$

The problem of making this result rigorous is solved for the cases $\varphi(t) \in L_1$, $\varphi(t) \in L_2$, and $\varphi(t)$ of bounded variation. [The author does not make any connection with the Poisson summation formula, to which his results are related.] *H. Pollard* (New York, N. Y.).

Borgen, S. Note on the summability of Poisson's formula. J. London Math. Soc. **19**, 100–105 (1944). [MF 12556]

Poisson's formula is established in the following general form. Let $f(x) \in L$ over any finite interval, let

$$(1) \qquad F(x) = (2/\pi)^{\frac{1}{2}} \int_0^{+\infty} f(t) \cos xt dt,$$

and let

$$(2) \qquad \chi(t) = \sum_{n=1}^{\infty} f(t+n\alpha), \qquad \alpha > 0,$$

the convergence in (2) being dominated in $-\frac{1}{2}\alpha \leqq t \leqq \frac{1}{2}\alpha$. Then, if $\alpha\beta = 2\pi$,

$$(3) \quad \lim_{\delta \to 0} \beta^{\frac{1}{2}} \left\{ \tfrac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n\beta) \varphi(\delta n \beta) \right\} = \tfrac{1}{2} \alpha^{\frac{1}{2}} \{ f(0+) + \chi(0+) + \chi(0-) \}$$

whenever the right side exists; $\varphi(u)$ is a rather general function, for example, e^{-u^2}, e^{-u}, or $1-u$ for $0 < u < 1$, 0 for $u > 1$. The hypothesis on (2) ensures the existence of (1) for the arguments involved in (3). *R. P. Boas, Jr.*

MacRobert, T. M. Expansions in terms of associated Legendre functions. Philos. Mag. (7) **35**, 670–680 (1944). [MF 11957]

An expansion of the form

$$f(z) = \sum_{n=0}^{\infty} A_n (z^2-1)^{-\frac{1}{4}m} P_{n+\frac{1}{2}}^{-m}(z) + \sum_{n=0}^{\infty} B_n (z^2-1)^{\frac{1}{4}m} Q_{n+\frac{1}{2}}^{m}(z)$$

is obtained for any function $f(z)$ holomorphic within and on the boundary of the region between two confocal ellipses with foci at ± 1. The special case of the expansion inside a single ellipse is also discussed. *M. C. Gray.*

Lévy, P. Sur une généralisation des fonctions orthogonales de M. Rademacher. Comment. Math. Helv. **16**, 146–152 (1944).

The author generalizes the Rademacher functions by defining a sequence of functions $\varphi_n(x)$ as follows. Let p be any prime, α a primitive root of $y^p = 1$. Put $\omega(x) = \alpha^{[x]}$, $\omega_k(x) = \omega(p^k x)$. Let $n = \epsilon_1 + \epsilon_2 p + \cdots$, $0 \leqq \epsilon_r < p$. Then

$$\varphi_n(x) = \omega_1^{\epsilon_1}(x) \omega_2^{\epsilon_2}(x) \cdots.$$

The author shows that

$$\int_0^1 \varphi_n^2(x) dx = 1, \quad \int_0^1 \varphi_n(x) \bar{\varphi}_m(x) dx = 0, \qquad n \neq m.$$

P. Erdös (Stanford University, Calif.).

Romanov, N. P. On a complete orthonormal system of the space $L_2(0, 1)$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 278–279 (1944). [MF 12667]

Put

$$F(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x < 1. \end{cases}$$

Let n run through the odd integers. Put $T_n(x) = nF(nx)$. The orthogonalization of $\{T_n(x)\}$ leads to the sequence $\{\rho_n(x)\}$, where

$$\rho_n(x) = (1/\varphi_2(n)) \sum_{d|n} \mu(n/d) T_d(x),$$

with

$$\varphi_2(n) = n^2 \prod_{p|n} (1-p^{-2}).$$

Using this system, P. S. Hagleev obtains several identities. *P. Erdös* (Stanford University, Calif.).

Bourgin, D. G., and Mendel, C. W. Orthonormal sets of periodic functions of the type $\{f(nx)\}$. Trans. Amer. Math. Soc. **57**, 332–363 (1945). [MF 12560]

The authors study a number of problems arising from the question of what odd real functions $f(x)$ of $L^2(-\pi, \pi)$ can satisfy

$$(1) \qquad \pi^{-1} \int_{-\pi}^{\pi} f(nx) f(mx) dx = \delta_{nm}, \quad n, m = 1, 2, \cdots.$$

The only complete set $\{f(nx)\}$ satisfying (1), or even the weaker condition with $\inf(n, m) \leqq 1$, arises from $f(x) = \pm \sin x$; for any other $f(x)$, the set $\{f(nx)\}$ cannot even be completed by adjoining a finite number of functions. Sharper results of similar character are given. Let $f(x) \sim \sum_{k=1}^{\infty} a_k \sin kx$. Then the condition that $f(x)$ satisfies (1) and $\sum |a_k| < \infty$ is equivalent to the following condition on $\varphi(z) = \sum a_k k^{-z}$: $\varphi(z)$ is meromorphic, the Dirichlet series defining $\varphi(z)$ converges absolutely for $\Re(z) \geqq 0$, and $\varphi(z)\varphi(-z) = 1$. Using this equivalence, the authors obtain solutions of (1) and discuss the construction of further solutions by combining them. They also discuss transformations of classes of solutions of (1); a wide variety of special conditions which force solutions to have special forms; topological properties in l^2 of the sequences $\{a_k\}$ corresponding to solutions of (1); and a theorem on singularities of power series associated with the problem. *R. P. Boas, Jr.*

Agnew, Ralph Palmer. Spans in Lebesgue and uniform spaces of translations of step functions. Bull. Amer. Math. Soc. **51**, 229–233 (1945). [MF 12258]

Let $F(x)$ be a simple step-function in $-\infty<x<\infty$, say $F(x)=h\neq 0$ in $-a\leqq x\leqq a$, $F(x)=0$ outside. Then, if $\int_{-\infty}^{\infty}|f(x)|^p dx<\infty$ for $p>1$, there exist, for every positive ϵ, constants $c_1, \cdots, c_n$ and $\gamma_1, \cdots, \gamma_n$ such that

$$\int |f(x)-\gamma_1 F(x+c_1)-\cdots-\gamma_n F(x+c_n)|^p dx<\epsilon.$$

If $g(x)$ is continuous and $g(x)\to 0$ as $x\to\pm\infty$, then we can have

$$|f(x)-\gamma_1 F(x+c_1)-\cdots-\gamma_n F(x+c_n)|<\epsilon$$

in $-\infty<x<\infty$. The second theorem is contained in a result of Boas and Bochner [Ann. of Math. (2) **39**, 287–300 (1938)]; however, the author's proofs are exceedingly simple. *S. Bochner* (Princeton, N. J.).

Natanson, I. P. On some estimations connected with singular integral of C. de la Vallée-Poussin. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 274–277 (1944). [MF 12666]

Let

$$V_n(x)=\frac{1}{2\pi}\frac{(2n)!!}{(2n-1)!!}\int_{-\pi}^{\pi} f(t)\cos^{2n}\tfrac{1}{2}(t-x)dt$$

be the singular integral of de la Vallée Poussin. It is proved that $n\{V_n(x)-f(x)\}\to f''(x)$ whenever $f''(x)$ exists, and that for continuous $f(x)$ one has $|V_n(x)-f(x)|\leqq 3\omega(1/\sqrt{n})$, where $\omega(\delta)$ is the modulus of continuity of $f(x)$. The author also establishes the result $|B_n(x)-f(x)|\leqq\frac{3}{2}\omega(1/\sqrt{n})$, where $B_n(x)$ is the Bernstein polynomial. This was first proved by T. Popoviciu [Mathematica, Cluj **10**, 49–54 (1935)] and then by a different method by the reviewer [Studia Math. **8**, 170 (1939); **7**, 49–51 (1938)]. The author's claim that his proof is considerably simpler than that of the reviewer might be misleading. Actually, the proofs are identical, except that the author avoids using the terminology of the theory of probability. [The estimates cited here for Bernstein polynomials are best possible; weaker estimates were obtained by S. Wigert [Ark. Mat. Astr. Phys. **20A**, no. 5 (1927)] by a rather laborious method.] *M. Kac* (Ithaca, N. Y.).

Bary, Nina. Sur les systèmes complets de fonctions orthogonales. Rec. Math. [Mat. Sbornik] N.S. **14(56)**, 51–108 (1944). (French. Russian summary) [MF 12292]

Sections 1 to 6 are devoted to the following problem. Let $\{x_n\}$, $\{y_n\}$ be sequences in Hilbert space, of which the first is complete and orthonormal and such that $d^2=\sum\|x_n-y_n\|^2<\infty$. What additional hypotheses on $\{y_n\}$ will guarantee that it also forms a complete set? The results have already been announced in an earlier abstract [C. R. (Doklady) Acad. Sci. (N.S.) **37**, 83–87 (1942); these Rev. **4**, 272]. [Attention is called to theorem 4 to which the additional hypothesis "$\psi_n(x)$ normalized" should be added. Theorem 5 states that, if $d<1$ and the $\{y_n\}$ are normalized, then the $\{y_n\}$ are complete. The condition that the $\{y_n\}$ are normalized is in fact superfluous. This is a consequence of a theorem of Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, p. 100] which establishes even more: if $d<1$, the $\{y_n\}$ form a basis.] Section 7 refines some results obtained in an earlier paper [same Rec. N.S. **12(54)**, 3–27 (1943); these Rev. **5**, 5]. *H. Pollard.*

Eichler, Martin. Eine Verallgemeinerung des Rungeschen Satzes. Math. Z. **49**, 565–575 (1944). [MF 11986]

The author proves the following theorem. (1) Let G and K be two simply-connected regions in n-dimensional Euclidean space, bounded by smooth finite surfaces and such that $G\subset K$. (2) Let the coefficients of the differential equation

$$L[u]=\sum_{i,k=1}^{n} A_{ik}u_{x_ix_k}+\sum_{j=1}^{n} B_j u_{x_j}+Cu=0$$

be analytic functions of $x_1, \cdots, x_n$ in K. Let $L[u]$ be of elliptic type and suppose that $A_{ik}\geqq 0$, $C\leqq 0$. (3) Let $u_1(X), u_2(X), \cdots$ (where $X=(x_1, \cdots, x_n)$) be a system of linearly independent solutions of $L[u]=0$ in K, which are integrable and complete (when orthogonalized) on the boundary of K. (4) Let the Green's function exist for K and for all regions having smooth boundaries and lying in K. Then: (I) the system $u_1(X), u_2(X), \cdots$ is also complete on the boundary g of G; (II) if the $u_i(X)$ are orthogonalized on g to the system $q_i(X)$, and if a solution $u(X)$ of $L[u]=0$ which is regular in G and integrable on g has on g the Fourier series $u(X)=\sum_{i=1}^{\infty}\alpha_i q_i(X)$, then the partial sums $\sum_{i=1}^{n}\alpha_i q_i(X)$ converge to $u(X)$ uniformly in every closed subregion of G. *L. H. Loomis* (Cambridge, Mass.).

Mandelbrojt, S. Dirichlet series. Rice Inst. Pamphlet **31**, 159–272 (1944). [MF 11836]

This monograph consists of a series of lectures delivered by the author and is not a complete treatment of general Dirichlet series. It is devoted instead to certain special topics which have interested the author and to which he has contributed many papers. Apart from preliminary details, the subject can be divided into three parts.

First, the nature of functions $f(s)=\sum a_n e^{-\lambda_n s}$, where the λ_n are subjected to conditions of an arithmetical character, is discussed. As an example, one can mention the following result, akin to the classical theorem that an entire function takes values everywhere dense in the plane. Let $D(\lambda)$, $0<\lambda<\frac{1}{2}$, be the region containing the positive real axis, of which the boundary is the curve

$$\sigma=\lambda\sin^2\tfrac{1}{4}(t-[t/2\pi]2\pi).$$

Suppose $\{\lambda_n\}$ has the property that the sequence consisting of all the λ_n and unity is composed of linearly independent elements (that is, each n of them, for all n, are linearly independent). If the series has a finite abscissa of absolute convergence, then the values which $f(s)$ assumes when $s\in D(\lambda)$ are everywhere dense in a circle with center at the origin and radius $A=$l.u.b. $|a_n|$.

Second, the properties of functions represented by Dirichlet series are discussed under the hypothesis that the λ_n have a finite upper density $D=\limsup n/\lambda_n$. This represents a generalization of theorems concerning power series whose coefficients have finite Pólya maximum density. In the passage from power series to Dirichlet series the domain of definition changes from a circle to a horizontal strip. A second step in generalization is taken by the author in passing from a horizontal strip to the "curvilinear" strip. Finally, for many of the results it is not even required that the function be represented by the series in the usual sense, but only in a certain asymptotic sense. The following theorem is an illustration. Actually a much deeper result is proved, but it is difficult to do it complete justice in a review. If $f(s)$, represented by a Dirichlet series, is entire, if $\liminf(\lambda_{n+1}-\lambda_n)>0$ and if $f(s)$ is of infinite Ritt order, then

in every horizontal strip of width $2\pi a$, $a>D$, there exists a Julia line.

The third topic treated is the extension to Dirichlet series of results in power series relating to the composition of singularities. These fall into two general types, the classical Hadamard theorem on the multiplication of singularities and the less widely known Hurwitz theorem: if E_1, E_2, E_3 are, respectively, the sets of singularities of the functions $\sum a_n z^{-n-1}$, $\sum b_n z^{-n-1}$, $\sum c_n z^{-n-1}$, where $c_n=\sum_0^n C_n a_i b_{n-i}$, and if $z_3 \varepsilon E_3$, then there exist z_1 in E_1 and z_2 in E_2 such that $z_3=z_1+z_2$. The generalization to Dirichlet series encounters difficulties from the start, for the "obvious" analogues are false. For the correct theorems the reader must be referred to the monograph itself. *H. Pollard* (New York, N. Y.).

Brunk, H. D. Theorems of composition for Dirichlet series. Duke Math. J. **12**, 1–21 (1945). [MF 12065]

Hadamard's theorem on the composition of singularities of functions defined by power series may be stated broadly as follows. If $\lambda_n=n$ and $f(s)=\sum a_n e^{-\lambda_n s}$, $\varphi(s)=\sum b_n e^{-\lambda_n s}$, $H(s)=\sum a_n b_n e^{-\lambda_n s}$, then the only singularities of the function $H(s)$ are of the form $\alpha+\beta$, where α and β are singularities of the functions $f(s)$ and $\varphi(s)$, respectively. This result is false if λ_n is an arbitrary increasing sequence, as is shown by the example

$$f(s)=\varphi(s)=H(s)=\zeta(s)=\sum n^{-s},$$

for which $\lambda_n=\log n$, $a_n=b_n=1$, as the only singularity of $H(s)$ is 1, which is not equal to $1+1$.

Mandelbrojt [Bull. Soc. Math. France **57**, 78–103 (1929); Acta Math. **55**, 1–32 (1930)] has given generalizations of Hadamard's theorem suitable for Dirichlet series. [For the most general of these see V. Bernstein, Leçons sur les Progrès Récents de la Théorie des Séries de Dirichlet, Gauthier-Villars, Paris, 1933, and the paper of Mandelbrojt reviewed above.] One of Mandelbrojt's theorems imposes conditions of boundedness on $f(s)$ and $\varphi(s)$ in certain domains and gives a domain of holomorphism of $H(s)$ in terms of the singularities of $f(s)$ and $\varphi(s)$. The author, assuming that the series for $f(s)$ and $\varphi(s)$ have abscissas of absolute convergence and that $f(s)$ and $\varphi(s)$ satisfy similar boundedness conditions, obtains a domain of holomorphism of $H(s)$ in terms of the domains in which $f(s)$ and $\varphi(s)$ are bounded. He also extends his result by assuming, instead of the boundedness conditions, that $f(s)$ and $\varphi(s)$ simply omit certain values in certain domains. He remarks, as an example, that the following known theorem is an easy consequence of his results. The function $\zeta(s)=\sum n^{-s}$ takes every value except at most one infinitely often, in every strip $\frac{1}{2}-\delta<\sigma<1-\delta$, for every positive δ. *L. S. Bosanquet.*

Ríos García, Sixto. On the rearrangement of series of functions and its applications. Abh. Math. Sem. Hansischen Univ. **15**, 57–81 (1943). (Spanish) [MF 12792]

Identical with a paper which appeared in Revista Mat. Hisp.-Amer. (4) **3**, 100–128 (1943); these Rev. **5**, 66.

Caton, W. B., and Hille, E. Laguerre polynomials and Laplace integrals. Duke Math. J. **12**, 217–242 (1945). [MF 12594]

Two principal subjects are considered: the Abel summability of Laguerre series, and the representation of functions analytic in a half-plane as Laplace integrals. The following special case illustrates the general character of the results on summability. If $F(u)$ is integrable in $(0, R)$ for all $R>0$, and $0\leqq c\leqq\frac{1}{2}$, where $c=\limsup_{u\to\infty} u^{-1}\log|\int_0^u F(t)dt|$, then the Laguerre series of $F(u)$ is Abel summable to it almost everywhere. As for the representation problem, it is proved first that any function analytic in $\Re(z)>\frac{1}{2}$ is the limit of a sequence of Laplace-Stieltjes integrals multiplied by a power of z. The representation problem for functions of $H_p(\frac{1}{2})$ is solved for the two cases $1<p\leqq 2$, $p>2$, the solution taking a different form in each case. In the course of the paper the following result is proved. If $g(y)\varepsilon L_p(-\infty, \infty)$ for some $p>1$, then its Fourier expansion in terms of the orthonormal set of functions

$$(2\pi)^{-\frac{1}{2}}(-\tfrac{1}{2}+iy)^n(\tfrac{1}{2}+iy)^{-n-1}$$

converges to it in the pth mean. This adds one set to the rather small number for which the p-mean convergence property has been established. *H. Pollard.*

Vignaux, J. C. On the asymptotic representation of functions by integrals. An. Soc. Ci. Argentina **138**, 27–39, 97–119, 249–260 (1944). (Spanish) [MF 12136]

The author says that $f(z)$ is represented asymptotically in $\Re(z)>0$ by the formal Laplace integral $\int_0^\infty e^{-zt}\varphi(t)dt$ if, for each positive number a,

$$e^{az}\left\{f(z)-\int_0^a e^{-zt}\varphi(t)dt\right\}\to 0$$

as $\Re(z)\to\infty$. For this asymptotic representation he establishes many theorems on the validity of formal operations, analogous to standard theorems for convergent integrals; for example, the representation is unique (for continuous $\varphi(t)$); the product of two functions $f(z)$ corresponds to the convolution of the associated φ's. These results are later extended to asymptotic Laplace-Stieltjes integrals. Replacing the Laplace integral by an asymptotic Laplace integral in the definition of Borel summability of a series, the author obtains a definition of asymptotic Borel summability, whose properties he investigates. A number of theorems, mostly known, on convergent, summable and oscillating Laplace integrals are also included. *R. P. Boas, Jr.*

Vignaux, J. C. Asymptotic series and integrals. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 180, Vol. **3**, num. 4. Serie segunda, 14, Contribuciones, 401–412 (1944). (Spanish) [MF 12191]

The concept of representation of $f(s)$ by an asymptotic Laplace integral $\int_0^\infty e^{-zt}\varphi(t)dt$ was defined in the paper reviewed above. The author shows that, if $\varphi(t)$ has derivatives of all orders, then $f(s)$ has an asymptotic power series and asymptotic factorial series in the classical sense. He also gives some results connecting Borel summability with asymptotic series. *R. P. Boas, Jr.* (Providence, R. I.).

Vignaux, J. C., and Cotlar, M. Asymptotic Laplace-Stieltjes integrals. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. No. 180, Vol. **3**, num. 4. Serie segunda, 14, Contribuciones, 345–400 (1944). (Spanish. French summary) [MF 12190]

In this paper classical results concerning Laplace integrals and asymptotic power series are extended to functions which are represented asymptotically by Laplace integrals. [Cf. the two preceding reviews.] Many classical results on inversion of and representation by a Laplace integral and more recent researches based on the Post-Widder inversion formula carry over to these asymptotic representations. Problems connected with asymptotic series are also generalized, for example, the following theorem of Borel-Carleman. Let $\{a_n\}$ be a sequence and D the region defined by

$0<|z|<1$, $-\alpha\pi<\arg z<\alpha\pi$, $\alpha<\infty$. Then there exists a function $f(x)$, analytic in D, such that

$$\lim x^{-n}\left[f(x)-\sum_0^n a_\nu x^\nu\right]=0$$

as $x\to 0$ through values in D. *H. Pollard.*

Alessi, Juan M. On the asymptotic integrals of LeRoy. An. Soc. Ci. Argentina **138**, 193–200 (1944). (Spanish) [MF 12137]

A function $f(z)$ is said by the author to be represented asymptotically by the integral

$$\int_0^\infty \phi(x)(1+xz)^{-1}dx$$

if for any $a>0$ it is true that

$$az\left[f(z)-\int_0^a \phi(x)dx/(1+xz)\right]$$

tends to zero as $\Re z\to\infty$. The author discusses the simplest properties of such representations, without explicit statement of the conditions under which his results are valid.

H. Pollard (New York, N. Y.).

Alessi, Juan M. On the transformation of LeRoy for a dual variable. An. Soc. Ci. Argentina **139**, 3–12 (1945). (Spanish) [MF 12592]

The transformation is

$$(1)\qquad f(z)=\int_0^1 \varphi(t)(1+tz)^{-1}dt,$$

where z is a "dual variable" ($z=x+ky$, $k^2=0$). The author verifies that (1) possesses a number of properties which are suggested by calculating formally as if z were an ordinary complex variable. *R. P. Boas, Jr.* (Providence, R. I.).

***Wagner, Karl Willy. Operatorenrechnung nebst Anwendungen in Physik und Technik.** J. W. Edwards, Ann Arbor, Michigan, 1944. xiv+448 pp. $9.50.

[Photographic reproduction of the original, which was published by J. A. Barth, Leipzig, 1940.] This book is addressed primarily to engineers and applied physicists; the author is mainly interested in discussing the various types of problems which can be solved with the aid of the operational calculus. The book opens with a brief introduction to the Heaviside operational calculus. This is followed by a study of the Laplace transform and its various properties which are useful in the solution of linear ordinary and partial differential equations. The Laplace transform is then used to solve numerous differential equations which arise in particle dynamics, and in the theory of electric circuits and transmission lines; the problems are of types which are often encountered in practice. Many of the problems are solved in numerical detail. A table of Laplace transforms is included. *A. E. Heins* (Cambridge, Mass.).

Shastri, N. A. Some theorems in operational calculus. Proc. Indian Acad. Sci., Sect. A. **20**, 211–223 (1944). [MF 12204]

If

$$f(p)=p\int_0^\infty e^{-ps}h(s)ds$$

and

$$h(s)\phi(s)=s\int_0^\infty e^{-st}g(t)dt$$

($\phi(s)$ assumed known), then there is an integral relation connecting $f(p)$ and $g(t)$. On the basis of this integral theorem, for special forms of $\phi(s)$, the author evaluates some integrals involving Bessel functions, Sonine polynomials, etc.

A. E. Heins (Cambridge, Mass.).

Colombo, Serge. Sur quelques nouvelles correspondances symboliques. Bull. Sci. Math. (2) **67**, 104–108 (1943). [MF 12633]

The author discusses the properties of the functions

$$v(x)=\int_0^\infty \frac{x^t dt}{\Gamma(t+1)},\qquad \frac{x^{\lambda-1}}{1+x^\lambda},\qquad \lambda>0,$$

by means of the unilateral Laplace transform. [Cf. C. R. Acad. Sci. Paris **216**, 368–369 (1943); these Rev. **5**, 238.]

A. E. Heins (Cambridge, Mass.).

Krein, M. On the logarithm of an infinitely decomposible Hermite-positive function. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 91–94 (1944). [MF 12572]

Krein, M. On the problem of continuation of helical arcs in Hilbert space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **45**, 139–142 (1944). [MF 12578]

I. J. Schoenberg and later Schoenberg and von Neumann have investigated, in $-\infty<x<\infty$, functions $g(x)$ of the special type

$$(*)\qquad g(x)=\alpha-\int_0^\infty \lambda^{-2}\sin^2\lambda x(1+\lambda^2)d\tau(\lambda),$$

where $\tau(\lambda)$ is bounded and nondecreasing. A function $g(x)$ is of the type (*) if the function $f(x)=e^{g(x)}$ is infinitely decomposable positive-definite, that is if $(f(x))^{1/n}$ is positive definite for $n=1, 2, 3, \cdots$. The functions (*) also play a role in the study of helical curves. A helical curve ξ_t is a mapping from the real line $-\infty<t<\infty$ into Hilbert space, for which the inner product $B(s,t)=(\xi_{r+s}-\xi_r, \xi_{r+t}-\xi_r)$ is independent of r.

The author has previously extended the theory of positive-definite functions from the infinite interval $-\infty<x<\infty$ to the arbitrary interval $-a<x<a$, $0<a<\infty$. In the present papers he makes a similar study for infinitely decomposable functions and for helical arcs, the latter being defined in $-a<t<a$. He shows that the formula (*) remains valid, and that if $g(x)$ is so representable in $(-a, a)$, then $g(x)$ can be continued to a function so representable in $(-\infty, \infty)$. Also, there are criteria for recognizing whether the continuation is unique or not. Furthermore, the author establishes the connection between helical arcs and functions $g(x)$, and he has theorems on the continuation of helical arcs into entire curves. The details connect with a theory of generalized moments which is to appear in book form. *S. Bochner* (Princeton, N. J.).

Krein, M. On a remarkable class of Hermitian operators. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 175–179 (1944). [MF 12054]

Continuing his preceding notes on generalizations of the moment problem [same C. R. (N.S.) **43**, 323–326 (1944); **44**, 131–134 (1944); these Rev. **6**, 131, 179] the author now elaborates a generalization of which the following is the most pertinent special case. Instead of the family of polynomials consider for fixed positive h the family of functions

$$\varphi(t)=\int_{-h}^{h} e^{itx}d\alpha(x),$$

where $\alpha(x)$ is a complex-valued function of bounded varia-

tion, and close the family to form a Hilbert space with the inner product

$$(\varphi, \psi) = \int_{-h}^{h}\int_{-h}^{h} F(x-y)d\alpha(x)d\bar{\beta}(y),$$

where $F(x)$ is any fixed function which can be written in the form $F(x)=\int_{-\infty}^{\infty}e^{ixt}d\sigma(t)$, with a bounded nondecreasing function $\sigma(t)$. Since $F(x)$ is needed only in $(-2h, 2h)$, the function $\sigma(t)$ may be "indeterminate." In the case of indeterminacy the kernel

$$G(z)=\sum|\varphi_k(z)|^2=\sup_f|(f, f)|/|f(z)|^2$$

is again finite in the complex z-plane, but the familiar estimate $O(e^{\delta(r)})$ is now to be replaced by a precise formula which, in particular, implies $G(z)=O(\exp\{2h|y|+r\delta(r)\})$. Perhaps the most interesting further conclusion from the general theory is the fact that the totality of all $\sigma(t)$ by which $F(x)$ can be represented is again involved in a formula of the form

$$\frac{p_0(z)+\tau(z)p_1(z)}{q_0(z)+\tau(z)q_1(z)}=\int_{-\infty}^{\infty}(t-z)^{-1}d\sigma(t),$$

in which the functions on the left are analytic functions similar to those in the classical theory. *S. Bochner.*

Krein, M. On a generalized problem of moments. C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 219–222 (1944). [MF 12034]

M. Livshitz has generalized the moment problem on $(-\infty, \infty)$ by replacing the family of polynomials by a more general linear space L of functions each of which is analytic in a neighborhood of the real axis; L has, more or less, the property that if $f(z)$ belongs to L then so do $zf(z)$ and $\{f(z)-f(a)\}/(z-a)$. Livshitz proved that corresponding to any "inner product" $B(f, g)$ with the property $B(zf, g)=B(f, zg)$ there exists at least one distribution function $\sigma(z)$ such that

$$B(f, g)=\int_{-\infty}^{\infty}g(t)f(t)d\sigma(t).$$

Krein amplifies this by announcing a great many analogues and adaptations of the classical moment theory. First, if the distribution function is not unique, then each $f(z)$ is meromorphic and its growth can be estimated. Furthermore, in that case, for any orthonormal sequence, the sum $\sum|\phi_k(z)|^2$ converges at every nonreal point in a fixed neighborhood of which all $f(z)$ are analytic. Every distribution function $\sigma(z)$ may be obtained from a relation of the form

$$\int_{-\infty}^{\infty}(t-z)^{-1}d\sigma(t)=\frac{p_0(z)+\tau(z)p_1(z)}{q_0(z)+\tau(z)q_1(z)}$$

in the customary fashion, and if $\tau(z)\equiv C$, then $\sigma(t)$ has derivative 0 almost everywhere. This is only a selection of the results announced. *S. Bochner* (Princeton, N. J.).

Pinney, Edmund. A class of integral equations which generalize Abel's equation. Bull. Amer. Math. Soc. **51**, 259–265 (1945). [MF 12263]

Let $g(x)$, $p(x)$, $q(x)$ be given functions, with $g'(x)$ continuous, such that

$$\int_0^{\infty}g(t)tdt,\quad \int_0^{\infty}\int_0^{\infty}|g((x^2+s^2+t^2)^{\frac{1}{2}})p(s)q(s)|dsdt$$

exist, and

$$(1)\qquad \int_0^{\pi/2}p(r\cos\theta)q(r\sin\theta)d\theta=1.$$

The author shows that the integral equation

$$(2)\qquad g(x)=\int_0^{\infty}\phi((x^2+t^2)^{\frac{1}{2}})p(t)dt$$

is solved by

$$-x\phi(x)=(d/dx)\int_0^{\infty}g((x^2+t^2)^{\frac{1}{2}})q(t)dt.$$

Examples of functions p and q satisfying (1) are given; a special choice turns equation (2) into Abel's integral equation. *F. G. Dressel* (Durham, N. C.).

Difference Equations, Special Functional Equations

Sternberg, Wolfgang J. A new solution for linear difference equations. Bull. Amer. Math. Soc. **51**, 462–464 (1945). [MF 12529]

The author shows that a solution of the difference equation $F(t+1)-F(t)=\phi(t)$ is

$$F(t)=-\tfrac{1}{2}\phi(t)+\int_a^t\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_a^t\phi(\tau)\cos 2\pi k(t-\tau)d\tau,$$

where a is a constant and $\phi(t)$ is continuous and of bounded variation in every finite interval. If $\phi(t)$ is discontinuous, the term $-\frac{1}{2}\phi(t)$ in the solution is replaced by $-\frac{1}{2}\phi(t-0)$. *A. E. Heins* (Cambridge, Mass.).

Hornstein, M. Sur les équations linéaires aux différences finies. Rec. Math. [Mat. Sbornik] N.S. **14** (**56**), 269–302 (1944). (Russian. French summary) [MF 12308]

Using a new method, the author presents developments related to his earlier work, as well as to that of Poincaré, Perron, Pincherle and Nörlund. The equation studied is

$$(A)\qquad D_{\nu+n}+a_1^{(\nu)}D_{\nu+n-1}+\cdots+a_n^{(\nu)}D_\nu=0.$$

A function D_ν is said to be regular and to belong to p if $\lim D_{\nu+1}/D_\nu=p$ is finite. An equation (A) all of whose solutions are regular is regular; otherwise it is irregular. If all the solutions are irregular, the equation is completely irregular. If (A) has regular and irregular solutions, it is semi-irregular. The following old results are proved anew. There exist completely irregular equations of Poincaré type, the moduli of whose characteristic equations are equal. There exist semi-irregular equations of Poincaré type whose characteristic equations have multiple roots. Some of the new results are as follows. There exist regular equations, the moduli of whose characteristic equations are equal. There exist regular equations for which the limits $\lim_\nu a_i^{(\nu)}$ do not exist. When (A) is regular, the solutions are grouped into n classes, belonging to $p_1, \cdots, p_n$ ($|p_1|\geqq\cdots\geqq|p_n|$), so that, when $D_\nu^{(j)}$ denotes any solution of the jth class, $\lim_\nu D_\nu^{(s)}/D_\nu^{(r)}=0$ for $r<s$. The author solves the problem of finding the initial conditions for the fundamental solutions of any regular equation. *W. J. Trjitzinsky.*

Fan, Ky. Sur le comportement asymptotique des solutions d'équations linéaires aux différences finies du second ordre. Bull. Soc. Math. France **70**, 76–96 (1942). [MF 11890]

This paper is concerned with solutions of a second order linear difference equation

$$\Delta^2y(n)+g(n)\Delta y(n)+f(n)y(n)=h(n)$$

which satisfy one of the following sets of conditions:

(I) $\lim_{n\to+\infty}[y(n)-n\Delta y(n)]$

exists;

(II) $\lim_{n\to+\infty}[y(n)-n\Delta y(n)]$

exists, and

$$\lim_{n\to+\infty} y(n)/n = \lim_{n\to+\infty} \Delta y(n) = 0;$$

(III) $\lim_{n\to+\infty} y(n)/n$ and $\lim_{n\to+\infty}\Delta y(n)$ exist and are equal. The results established, as well as the methods of proof used, are analogous to the results and methods of Haupt for a linear differential equation [Math. Z. 48, 212–220, 289–292 (1942); these Rev. 4, 276]. *W. T. Reid.*

Vietoris, L. Zur Kennzeichnung des Sinus und verwandter Funktionen durch Funktionalgleichungen. J. Reine Angew. Math. 186, 1–15 (1944). [MF 12096]

The following closely related functional equations for trigonometric functions are solved: (a) the system

$$C(x-\xi)=C(x)C(\xi)+S(x)S(\xi),$$
$$S(x-\xi)=S(x)C(\xi)-C(x)S(\xi);$$

(b) the system

$$C(x+\xi)=C(x)C(\xi)-S(x)S(\xi),$$
$$S(x+\xi)=S(x)C(\xi)+C(x)S(\xi);$$

(c) the d'Alembert-Poisson equation

$$C(x+\xi)+C(x-\xi)=2C(x)C(\xi);$$

(d) the equation

$$\{S(x)+S(\xi)\}\{S(x)-S(\xi)\}=S(x+\xi)S(x-\xi).$$

The solution of all four problems is reduced to that of the equation

(1) $$A(x+\xi)=A(x)A(\xi)$$

for a complex function $A(x)=\exp\{u(x)+i\varphi(x)\}$, where u and φ are real functions of the real variable x satisfying

(2) $$u(x+\xi)=u(x)+u(\xi),$$

(3) $$\varphi(x+\xi)\equiv\varphi(x)+\varphi(\xi) \pmod{2\pi}.$$

The general solution of (2) had been given by Hamel. A new solution of (3) is given here, which also makes use of a "Hamel base" for all real numbers, and is somewhat simpler than a previous solution by van der Corput. *F. John.*

Ascoli, Guido. Sopra un'equazione funzionale. Portugaliae Math. 4, 145–157 (1945). [MF 12445]

The equation is

(A) $$nf(x,ny)=f(x/n,y)+f((x+1)/n,y)+\cdots+f((x+n-1)/n,y),$$

which is satisfied by $\Psi(x+iy)-\log y$, where

$$\Psi(z)=\Gamma'(z)/\Gamma(z).$$

All solutions of (A) which are continuous in the half-strip S where $0\leqq x\leqq 1$, $y\geqq y_0>0$ have Fourier series, with respect to x, of the form

$$c_0+\sum_{n=-\infty}^{\infty}{}' a(ny)e^{2\pi inx},$$

where $\sum'$ omits the term with $n=0$. Alternatively, let $\beta_s(x,y)=y^{-s}B_s(x)$, B_s a Bernoulli polynomial. Then $f(x,y)$, continuous in S, satisfies (A) if and only if it can be uniformly approximated in S by a linear combination of the $\beta_s(x,y)$. In particular, for $\Psi(x+iy)-\log y$ we have $a(t)=e^{-2\pi t}\,\mathrm{Ei}\,(2\pi t)$. This leads to an asymptotic formula suitable for numerical calculation. The only solutions of (A) of the form $\Phi(x+iy)+\omega(y)$, with Φ analytic in S, are linear combinations of 1, $\Psi(x+iy)-\log y$, and $\cot\pi(x+iy)$. *R. P. Boas, Jr.* (Providence, R. I.).

Robinson, Lewis Bayard. A functional differential equation whose limit is the equation of velocity retardation. Revista Ci., Lima 46, 585–603 (1944). [MF 11969]

The author studies the analytic character of those solutions of the functional differential equation

$$u'(x)=a(x)\{u(x^p)\}^k$$

which are single valued at the origin. The exponents p and k are supposed to be rational and not less than unity. It is found that (1) a solution holomorphic at the origin is holomorphic in the interior of the unit circle; (2) (excluding certain special cases) if there is any singular point on the unit circle, the entire unit circle is singular and constitutes a frontier for the function $u(x)$. *W. E. Milne.*

Silberstein, Ludwik. Construction of groups of commutative functions. Philos. Mag. (7) 36, 43–54 (1945). [MF 12686]

The real functions $f(x)$ and $g(x)$ are said to be commutative if they satisfy the functional equation $f(g(x))=g(f(x))$. If $g(x)$ is an analytic function, this equation implies infinitely many equations for the Maclaurin coefficients a_n of $f(x)$, but these are not necessarily of a recursive nature. In the case $g(x)=e^x-1$, the first few equations suggest that one can take $a_0=0$, $a_1=1$, $a_2=a$ (arbitrary); the following coefficients then appear to become polynomials in a. The resulting functions $f_a(x)$ (should the series converge) appear to satisfy the group relation $f_af_b=f_{a+b}$. Similar remarks are made for $g(x)=\sin x$. *W. Feller* (Ithaca, N. Y.).

Integral Equations

Michlin, S. On the convergence of Fredholm series. C. R. (Doklady) Acad. Sci. URSS (N.S.) 42, 373–376 (1944). [MF 11643]

The linear homogenous integral equation of the second kind with finite integration range and kernel $K(x,s)$ has a resolvent $\Gamma(x,s;\lambda)=\Delta(x,s;\lambda)/\Delta\lambda$. The author's purpose is to give a new proof of the known theorem that the series in powers of λ for both numerator and denominator converge for all λ when $K\varepsilon L_2$. The argument hinges on the result (A) $\delta\lambda=\int_a^b\Gamma(s,s;\lambda)ds$ is analytic except for poles in $|\lambda|\leqq R$. This established, the observation that $\Delta\lambda$ is the integral function $\exp\int^\lambda\delta\lambda d\lambda$ and the use of the resolvent equation connecting K and Γ yields the final conclusion directly. [This, of course, has the slight disadvantage of requiring an independent argument to establish the existence and properties of Γ.] The result (A) is obtained by writing $K=K_N+K'$, where $K_N=\sum_1^N u_k(x)v_k(s)$, assuming that $\|K'\|<(R+1)^{-1}$ for suitable choice of K_N. It is known that Γ' is then analytic in $|\lambda|\leqq R$, hence the nature of (A) in this circle is determined by $\int_a^b(\Gamma-\Gamma')d\lambda$. *D. G. Bourgin.*

Azevedo do Amaral, Ignacio M. On integral equations. Anais Acad. Brasil. Ci. 16, 261–271 (1944). (Portuguese) [MF 12209]

The author extends his previous considerations [see the same Anais 16, 23–39 (1944); these Rev. 6, 157 and references given there] on particular types of solutions of linear

integral and ordinary differential equations. The results of these papers may be viewed as giving solutions in terms of "arbitrary" functions which satisfy certain functional relations with the kernel. From the discussion of the homogeneous Fredholm integral equation of the second kind in the present paper, however, it becomes evident that the author has overlooked the fact that he has required the "arbitrary" functions to satisfy such relations; indeed, for an integral equation of the first kind with fixed limits the derived integral formula for a solution is not needed, since the functional relation alone immediately determines a solution of the equation. He does show, however, that his methods may be used to determine the discontinuous solution $u(x)=kx^{\alpha-1}$ of the integral equation $u(x)=\int_0^x \alpha^{\alpha-\alpha}u(\alpha)d\alpha$.

W. T. Reid (Evanston, Ill.).

Azevedo do Amaral, Ignacio M. On the solution of Volterra's integral equation of the first kind. Anais Acad. Brasil. Ci. **16**, 273–277 (1944). (Portuguese) [MF 12210]

The author shows that his special methods may be used to obtain the well-known form of the solution of Abel's integral equation

$$F(x)=\int_0^x (x-\alpha)^{-n}u(\alpha)d\alpha, \qquad 0<n<1,$$

where $F(0)=0$. His more general discussion of the Volterra integral equation of the first kind suffers from the type of oversight mentioned in the preceding review.

W. T. Reid (Evanston, Ill.).

Fock, V. Sur certaines équations intégrales de physique mathématique. Rec. Math. [Mat. Sbornik] N.S. **14(56)**, 3–50 (1944). (Russian. French summary) [MF 12291]

The author studies the integral equation

$$(*) \qquad f(x)=g(x)+\int_0^\infty k(|x-y|)f(y)dy,$$

supposing that there exists a constant $c>0$ such that $e^{cx}k(x)$ is integrable and of bounded variation in $(0, \infty)$. Let $K(w)$ be the Fourier transform of $k(x)$ and suppose that $(*_*)$ $K(w)=1$ has no real roots. Then, (1) if v_0 is the imaginary part of that root of $(*_*)$ which is smallest in absolute value, the homogeneous equation (*) (with $g(x)\equiv 0$) has no solution $f(x)$ with $(*_*)$ $|f(x)|<e^{vx}L$, $v<v_0$. (2) If $g(x)$ is of bounded variation in $(0, \infty)$, (*) has a unique solution. (3) If $(*_*)$ has $2n$ real roots, the homogeneous equation has exactly n linearly independent solutions γ_r satisfying $(*_*)$; the nonhomogeneous equation has then a solution if and only if $g(x)$ is orthogonal to these γ_r and satisfies certain integrability conditions.

W. Feller (Ithaca, N. Y.).

Trjitzinsky, W. J. Integral equations in problems of representation of functions of a complex variable. Duke Math. J. **12**, 419–440 (1945). [MF 12613]

The problem is to find solutions $\varphi(J)$ of

$$\iint_K k(z, J)\varphi(J)dm(e_J)/(J-z)-f(z)=a(z),$$

where K is a suitable region and the kernel $k(z, J)$, the function $f(z)$ and the absolutely continuous additive set function $m(e)$ are given; $f(z)$ is, in general, not analytic; $a(z)$ is not given in advance, but is to be analytic. Under a variety of hypotheses on the classes to which $k(z, J)$ and $f(z)$ belong, the problem is shown to be equivalent to solving a regular Fredholm integral equation of the second kind. The simplest set of conditions requires that $f(z)\in \mathrm{Lip}\ 1$, uniformly in K; $K(z, J)\in L^2$ as a function of J; and $|K(z, J)-K(z', J)|\leqq|z-z'|\,\nu(J)$, $\nu(J)\in L^2$, where

$$K(z, J)=\{k(z, J)-k(z, z)\}/(J-z).$$

Other sets of conditions involve definitions from the author's theory of generalized analytic functions. Solutions of the problem lead to representations for classes of nonanalytic functions.

R. P. Boas, Jr. (Providence, R. I.).

Muschelišvili, N. I., and Vekua, N. P. Riemann's boundary value problem for several unknown functions and its application to systems of singular integral equations. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **12**, 1–46 (1943). (Russian. Georgian summary) [MF 11684]

The authors give a detailed account of a paper of J. Plemelj [Monatsh. Math. Phys. **19**, 211–246 (1908)] on the homogeneous Riemann boundary value problem for several unknown functions. Plemelj's results are used for the solution of the corresponding nonhomogeneous problem. The method requires the consideration of singular integral equations involving Cauchy's integral. According to the authors, their procedure is similar to but more general than that of G. Giraud [Ann. Sci. École Norm. Sup. (3) **56**, 119–172 (1939); these Rev. **1**, 145]. *A. Weinstein* (Toronto, Ont.).

Vekua, N. P. On the theory of systems of singular integral equations with kernels of Cauchy's type. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 207–214 (1943). (Russian. Georgian summary) [MF 11699]

The author considers a system of singular integral equations with Cauchy kernels (integration in the sense of Cauchy principal values over a set of simple closed "smooth" nonintersecting plane curves). The theory has been given in a complete form by N. I. Muschelišvili and the author [see the preceding review]. On the basis of a method applied by I. Vekua [same Bull. **3**, 869–876 (1942); these Rev. **5**, 268] to a single equation, the author solves the problem of equivalence of the system to equations of Fredholm type. Use is made of "regularizing" operators, an idea previously used by Muschelišvili.

W. J. Trjitzinsky (Urbana, Ill.).

Muschelišvili, N. I. Remark to the paper "Systems of singular integral equations with kernels of Cauchy's type." Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 99–101 (1943). (Russian. Georgian summary) [MF 11695]

The author indicates the extremely brief and simple proof due to the author and to N. P. Vekua of the theorem that the difference $l-l'$ of the numbers of solutions of the conjugate homogeneous equations $A\phi=0$, $A'\psi=0$ depends on the characteristic part of the operator A [see the author's paper, same Bull. **3**, 987–994 (1942); these Rev. **5**, 269]. The proof is based on the use of a "regularizing" operator A_1 such that A_1A or AA_1 is of Fredholm type.

W. J. Trjitzinsky.

Weinberg, M. M. On the existence of an Eigenfunktion for a class of non-linear integral equations. C. R. (Doklady) Acad. Sci. URSS (N.S.) **46**, 47–50 (1945). [MF 12546]

Using the characteristic values and functions of K, the author proves the following result. The equation

$$\lambda u(x)=\int_B K(x, y)g(u(y), y)dy,$$

where B is a closed bounded domain in Euclidean m-space and $g(0, x)=0$, has a solution $u\not\equiv 0$ for a positive λ if (I) K is symmetric, bounded and positive (in the terminology of integral equations); (II) $g(u, x)$ is continuous for x in B and $|u|<c$; (III) $ug(u, x)\geqq 0$ and $|g(u, x)|\geqq|f(u)|$ (f increasing on $(-c, c)$). *W. J. Trjitzinsky* (Urbana, Ill.).

Sobolev, V. On the intensity of radiation in the inner layers of absorbing and scattering medium. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **8**, 273–280 (1944). (Russian. English summary) [MF 12139]

Ambarzumian's integral equation

$$b(y)=\tfrac{1}{2}\lambda\int_{-1}^{1}\frac{p(y, y')b(y')}{1+ky'}dy' \tag{1}$$

arises in the theory of the absorption and scattering of radiation. In many cases of practical importance the kernel $p(y, y')$ is degenerate, and can be written

$$p(y, y')=\sum_{i=1}^{n}x_iP_i(y)P_i(y'),$$

where $P_i(y)$ is the ith Legendre polynomial. As is well known, equation (1) can then be solved explicitly. If n is not small, however, the numerical work becomes extremely laborious. The author shows that the process of solution can be simplified by writing

$$P_i(y)=(1+ky)\sum_{j=0}^{i-1}(2j+1)A_{ij}P_j(y)+C_i$$

and calculating A_{ij} and C_i once for all as functions of k. Tables and a numerical example are given.

F. Smithies (London).

Pitt, H. R. On a class of integro-differential equations. Proc. Cambridge Philos. Soc. **40**, 199–211 (1944). [MF 11866]

The author shows that, just as the general solution of the simple difference equation $f(x+c)-f(x)=0$ can (under very general conditions) be expressed in terms of the Fourier series

$$f(x)\sim\sum_{n=-\infty}^{\infty}a_ne^{2n\pi ix/c},$$

so the general solution (with certain weak restrictions at ∞) of the integro-differential equation

$$\sum_{r=0}^{R}\int_{-\infty}^{\infty}f^{(r)}(x-y)dk_r(y)=0 \tag{1}$$

can be expressed in terms of a "generalized Fourier series"

$$f(x)\sim\sum A_n(x)e^{\omega_nx} \tag{2}$$

in which the coefficients $A_n(x)$ are polynomials. The exponents in this series are the zeros of the function

$$K(\omega)=\sum_{r=0}^{R}\omega^r\int_{-\infty}^{\infty}e^{-\omega y}dk_r(y), \tag{3}$$

and the degree of each polynomial coefficient $A_n(x)$ is less than the order of the zero of $K(\omega)$ at $\omega=\omega_n$. (If $R=0$ and $k_0(y)$ is a step function with jumps of 1 and -1 at 0 and c, this reduces to the ordinary Fourier series.)

The hypothesis on $k_r(y)$ is that, for $-\infty<\sigma_1\leqq\sigma_2<\infty$, $\eta>0$,

$$\int_{-\infty}^{\infty}\{e^{-(\sigma_1-\eta)y}+e^{-(\sigma_2+\eta)y}\}\,|dk_r(y)|<\infty, \tag{4}$$

while the solutions $f(x)$ are to belong to $S_p(R, \sigma_1, \sigma_2, \alpha_1, \alpha_2)$ ($p\geqq 1$), which means that $f^{(r)}(x)$ is absolutely continuous ($r=0, 1, \cdots, R-1$) and

$$\left\{\sum_{r=0}^{R}|f^{(r)}(x)|\right\}\Big/\{(1+|x|^{\alpha_1})e^{\sigma_1x}+(1+|x|^{\alpha_2})e^{\sigma_2x}\}$$

is bounded in the Stepanoff sense with index p. Limits in an S-class are defined correspondingly. The author gives convergence theorems showing various senses in which solutions $f(x)$ of (1) which belong to some S-class are represented by their series (2). In particular, he shows that a function $f(x)$ of $S_p(R, \sigma_1, \sigma_2, \alpha_1, \alpha_2)$ is a solution of (1), in which $k_r(x)$ satisfies (4), if and only if it is the limit in $S_p(R, \sigma_1, \sigma_2, \alpha_1, \alpha_2)$ of finite linear combinations of the functions $A_n(x)e^{\omega_nx}$, where the ω_n are the zeros of $K(\omega)$, defined in (3), and the $A_n(x)$ are polynomials of degree less than the orders of the zeros. Finally, the author gives conditions under which the solutions $f(x)$ of (1) will be almost periodic in the Stepanoff or Bohr sense. *R. H. Cameron.*

Functional Analysis

Comét, Stig. Un problème de la géométrie à n dimensions. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **12**, no. 2, 14–24 (1942). [MF 12587]

In the Euclidean plane there are simple symmetric formulas which give the center and radius of the circle orthogonal to three pairwise tangent circles. Extending the results of an earlier note [same Proc. **11**, no. 9 (1941); these Rev. **3**, 86] the author gives analogous formulas for an n-dimensional linear space over a commutative field C, with an inner product defined by a symmetric nonsingular matrix of elements of C. Restrictions are needed on the characteristic of the field C and on the relative positions of the centers of the pairwise tangent spheres. *M. M. Day.*

James, R. C. Orthogonality in normed linear spaces. Duke Math. J. **12**, 291–302 (1945). [MF 12600]

The author gives three characterizations of orthogonality in a Euclidean space in terms of the norm and studies their individual virtues and vices as definitions of orthogonality in a general normed linear space. He shows that the three kinds of orthogonality may be distinct even in a two-dimensional space. The principal theorems show that if any one of these definitions gives too many of the ordinary properties of orthogonality, then the norm in the space must be definable from an inner product. *M. M. Day.*

Pontrjagin, L. Hermitian operators in spaces with indefinite metric. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 243–280 (1944). (Russian. English summary) [MF 12301]

The author considers the linear space H_k of all sequences $x=(x_1, x_2, \cdots)$ of complex numbers such that $\sum|x_n|^2<\infty$, with the inner product

$$(x, y)=-\sum_{i=1}^{k}x_i\bar{y}_i+\sum_{i=k+1}^{\infty}x_i\bar{y}_i.$$

Convergence and other purely topological notions are, however, defined in terms of the ordinary Hilbert space norm $\|x\|=(\sum|x_n|^2)^{\frac{1}{2}}$. The elements x, y are said to be orthogonal to one another if $(x, y)=0$; it may be noted that in this

space a nonnull element may be orthogonal to itself. The quadratic form (x, x) is said to be nondegenerate on the closed linear subspace $Q \subset H_k$ if the only element of Q orthogonal to every element of Q is $x=0$. It is shown that, if (x, x) is nondegenerate on Q, then Q is isomorphic to a space H_l with $l \leqq k$, (x, x) is nondegenerate on the orthogonal complement Q' of Q, and H_k is the direct sum of the orthogonal subspaces Q and Q'.

The operator A with domain Ω everywhere dense in H_k is said to be symmetric if $(Ax, y)=(x, Ay)$ for all $x, y \in \Omega$. The adjoint A^* of A is defined in the usual way, and A is said to be Hermitian if $A=A^*$. The space H_k is unlike ordinary Hilbert space in that a symmetric operator may have nonreal eigenvalues and nonsimple elementary divisors. The number and order of these are, however, limited by expressions depending on the integer k.

The main theorem states that every Hermitian operator A in H_k possesses an invariant subspace I of dimension k on which the quadratic form (x, x) is nonpositive, the corresponding eigenvalues all having nonnegative imaginary parts. Thus every Hermitian (self-adjoint) operator in H_k has a nonvoid point spectrum. An outline of the somewhat complicated proof is given in the English summary. The results are stated to have applications to problems arising in mechanics. *F. Smithies* (London).

Julia, Gaston. La théorie des fonctions et la théorie des opérateurs de l'espace hilbertien. J. Math. Pures Appl. (9) **22**, 71–83 (1943). [MF 12309]

The paper is largely a résumé of a number of the author's previously published results on linear operators in Hilbert space. He is concerned with the problems of characterizing the domains of existence of unbounded linear operators and of representing these operators throughout their domains by simple analytic means. He attempts to discuss these problems without recourse to the usual spectral representation, but rather by means of certain analogies he notes between the theory of functions of a complex variable and the theory of operators. He is not, however, completely able to do this, and in several cases bases his results upon theorems of von Neumann, Riesz and Lorch, which depend on spectral theory. *H. H. Goldstine* (Philadelphia, Pa.).

Julia, Gaston. Sur la convergence dans l'espace hilbertien. C. R. Acad. Sci. Paris **218**, 376–380 (1944). [MF 12109]

The author shows that a sequence X^n of points in a Hilbert space H converges strongly if and only if (X^n, Z) converges uniformly on the sphere $\|Z\|=1$. Another necessary and sufficient condition is that $\|X^n-Z\|^2-\|X^n\|^2$ converges uniformly on the unit sphere. In a previous paper [C. R. Acad. Sci. Paris **216**, 97–100 (1943); these Rev. **5**, 186] he showed that X^n converges weakly if in the preceding statement the uniformity is omitted. He next considers Landau's theorem concerning the convergence of $\sum|a_i|^2$. If $\sum|a_i|^2$ diverges, the domain D of convergence of $\sum \bar{a}_i x_i$, $(x_i=(e_i, X)$ with $\{e_i\}$ an orthonormal base for H) is a linear nonclosed set, everywhere dense in H. He considers the case where there is a linear closed set V in D, of infinitely many dimensions. If $\sum \bar{a}_i x_i$ converges for every X in V, then there is an α in V such that $\sum a_i e_i'$ converges to α, where e_i' is the projection of e_i on V. Hence the convergence to α is strong if and only if the series $\sum \bar{a}_i x_i$ converges uniformly on the unit sphere of V. If $V=H$ we have Landau's theorem. The paper then closes with two examples, in one of which $\sum a_i e_i'$ converges weakly and in the other strongly, on spaces $V \neq H$. *H. H. Goldstine* (Philadelphia, Pa.).

Marcouchevitch, A. Sur une généralisation d'une théorème de Menchoff. Rec. Math. [Mat. Sbornik] N.S. **15(57)**, 433–436 (1944). (Russian. French summary) [MF 12305]

Let E be a Banach space, $\{u_n\}$ a sequence of elements such that the closed linear manifold determined by $\{u_n\}$ is E; suppose that $\{u_n\}$ has a biorthogonal sequence $\{L_n\}$ of linear functionals. Then every element of E can be written as $x^{(1)}+x^{(2)}$ in such a way that the expansions $\sum L_n(x^{(i)})u_n$ have subsequences of partial sums converging to $x^{(i)}$ $(i=1, 2)$. [Cf. C. R. (Doklady) Acad. Sci. URSS. (N.S.) **44**, 262–264 (1944); these Rev. **6**, 179.] *R. P. Boas, Jr.*

Mackey, George W. On infinite-dimensional linear spaces. Trans. Amer. Math. Soc. **57**, 155–207 (1945). [MF 12129]

A "linear system" is equivalent to a linear topological space. However, one considers as given not a linear space X and a topology but rather a linear space X and the corresponding set of continuous linear functionals L. For the notion of a linear space X itself, the cardinal number of a Hamel basis is determining and is called the dimension of X. It is shown that for a complete normed space this is either finite or at least c. Nevertheless, the countably-dimensional linear space is of considerable interest from the point of view of the present paper and is frequently discussed.

The notion of isomorphism between linear systems is developed, based on the ideas of a linear transformation between the spaces and, essentially, of the adjoint between the corresponding sets of linear functionals. For a set M in X, M' is defined as the orthogonal complement of M in L; for M in L, M' is the intersection of the orthogonal complement of M with X. A set M is said to be closed if it is a prime set. Essentially, two linear systems are isomorphic if and only if the lattices of their closed subspaces are equivalent.

An ordered pair of elements b, c of a lattice is called modular if, for every $a \leqq c$, we have $(a \cup b) \cap c = a \cup (b \cap c)$. A pair a, b is called d-modular if we have this relationship for all $c \geqq a$. A pair of closed sets M and N is d-modular in the lattice of all closed subsets of either X or L of a linear system if and only if $M \cdot + \cdot N$ is closed. For a complete normed space modularity and d-modularity are equivalent, although not equivalent in general. Two closed subspaces M and N are defined to be "tangent" by means of a definition based on linear transformations; the definition is shown to be equivalent to the property that $M \cdot + \cdot N$ is not closed. The author shows that every infinite-dimensional closed linear space contains a pair of tangent subspaces. The modularity properties are also discussed relative to changes of M (or N) in which the new M, M^*, is such that the quotient space M^*/M or M/M^* is one-dimensional. If either modularity or d-modularity is unaffected by such a change, the pair is said to be a stable modular pair or a stable d-modular pair. Stable modularity is equivalent to stable d-modularity.

A set M of X is said to be bounded if, for every $l \in L$, the set of real numbers $l(x)$ $(x \in M)$ is bounded. Similarly, if $M \subset L$, it is said to be bounded if, for every $x \in X$, the set of $l(x)$ $(l \in M)$ is bounded. The bounded closure of L is the set of all linear functionals l on X which have the same bounded sets as L. A closed subspace in L is always boundedly closed but in general the boundedly closed subsets of L are not closed. Using this notion of bounded set in X, the convergence of a sequence of elements $x_1, x_2, \cdots$ to x can be defined as the property that there exists a sequence of real

numbers, $n_1, n_2, \cdots$, with $n_i \to \infty$, such that the set of elements of the form $n_i(x_i - x)$ is bounded. The derived set $M^{(1)}$ of a given set M is defined as the set of limit points of M; L is boundedly closed if and only if all subsets M of X for which the quotient space X/M is one-dimensional are sequentially closed. The notion of a bounded set can also be used to define a Cauchy sequence and hence completeness. The author shows that, if L is the set X^* of all linear functionals on X and the dimension of X is not too large, then the system in which $L = X^*$ is the linear space and X the set of linear functionals is boundedly closed.

The author next defines five properties of the set of all bounded subsets of either part of a system. These properties are studied relative to increases and decreases of the original space involving quotient spaces of various dimensionalities. A sixth property which involves simultaneously the bounded sets of X and L is also considered. These properties are related to the process of specializing a linear system to a normed set. Examples are given to show the existence of certain possibilities. The paper concludes with a set of unsolved problems. *F. J. Murray* (New York, N. Y.).

Hyers, D. H. Linear topological spaces. Bull. Amer. Math. Soc. **51**, 1–21 (1945). [MF 11766]

This address is a general report on linear topological spaces; that is, linear spaces which are at the same time topological spaces, and in which the fundamental operations of addition and scalar multiplication are continuous. The author discusses boundedness and convexity, normability and metrizability, fixed point theorems and functional equations, pseudo-norms, linear transformations and functionals, differentials, convergence and completeness. An application of a fixed point theorem to hydrodynamics is mentioned. [The author states that the method of fixed points leads only to existence theorems in the theory of functional equations, and tells nothing about the uniqueness of the solution. It seems that he has overlooked the criterion of uniqueness obtained by Leray and Schauder [Ann. Sci. École Norm. Sup. (3) **51**, 45–78 (1934)] by using Fréchet's differential. This criterion was established with view to an application to the Helmholtz problem of hydrodynamics [Leray, Comment. Math. Helv. **8**, 250–263 (1935), where the connection with Weinstein's results on uniqueness is given]. Weyl's paper [Ann. of Math. (2) **43**, 381–407 (1942); these Rev. **3**, 284], mentioned by the author, follows closely the methods of Leray and Schauder.] *A. Weinstein* (Toronto, Ont.).

Wehausen, John V. Transformations in metric spaces and ordinary differential equations. Bull. Amer. Math. Soc. **51**, 113–119 (1945). [MF 11825]

Let M be a complete metric space. The properly metrized class of all completely continuous transformations of M into itself is then a complete metric space. Let T be a subclass of this space and T_0, T_1, T_2 the subclasses of those transformations of T for which the number of fixed points in M is, respectively, 0, exactly 1, and at least two. The author proves first some theorems about these classes; for example, (1) T_0 is open; (2) if T_1 is dense in T, T_0 is empty and T_2 is of the first category in T; if T is complete, T_1 is of the second category in T; (3) if T_2 is dense in T and if A is a closed set in T contained in T_1, then A is nowhere dense in T. These theorems are then applied to the theory of a system of ordinary differential equations: the classical existence theorem for continuous differential equations appears as a consequence of (2) if it is admitted that the system $y_i = F_i(t, y_1, \cdots, y_n)$, $i = 1, \cdots, n$, has one and only one solution in the case of polynomials F_i. (The author remarks that without any such assumption the existence of a solution follows, as is well known, from Schauder's fixed point theorem, but that his theorems are much more elementary than the Schauder theorem.) Result (2) yields the information that those differential equation problems (considered in a specified way as points in T) which have multiple solutions are of the first category in T, while those with unique solutions are of second category, a fact closely related to a result of W. Orlicz [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. **8–9**, 221–228 (1932)]. Result (3) yields the fact that the class of problems satisfying a uniform Lipschitz condition (and another condition assuring uniqueness) is of first category in T, which is again related to a result of Orlicz. Another application deals with the continuous dependence of the solutions on parameters contained in the differential equations. *E. H. Rothe.*

Alexandroff, A. D. Additive set-functions in abstract spaces. Rec. Math. [Mat. Sbornik] N.S. **13(55)**, 169–238 (1943). (English. Russian summary) [MF 11651]

These are the concluding three chapters of a monograph of six chapters under the same title. The introduction and chapters 1–3 appeared in Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 307–348 (1940); **9(51)**, 563–628 (1941); cf. these Rev. **2**, 315; **3**, 207.

Chapter 4 is called "necessary and sufficient conditions for weak convergence of charges." A "charge" is a generalized measure; see the introduction for the definition. The principal theorems are these: a necessary and sufficient condition in a space R for the weak convergence of a sequence of charges $\mu_n(E)$ to a charge $\mu(E)$ is that $\lim \mu_n(R) = \mu(R)$, and $\mu(F) \geqq \limsup \mu_n(F)$ for every closed set F; in a completely normal space a necessary and sufficient condition for $\mu_n(E)$ to converge weakly to $\mu(E)$ is that for each closed set F and open set $G \supset F$ there exists a sequence of closed sets F_n such that (1) $F \subset F_n \subset G$, (2) for every n there is an m such that $F_n \supset F_{n+k}$ for $k \geqq m$, (3) $\prod F_n = F$, (4) $\lim \mu_n(F_n) = \mu(F)$.

Chapter 5 is devoted to "some theorems on weak convergence of charges." A pairwise disjoint sequence of closed sets F_n is called divergent if the sum of any number of the F_n is closed. A set of charges is said to contain an "escaping" or "eluding" load if it contains a sequence of charges $\mu_n(E)$ for which one can find a number $a \neq 0$ and a divergent sequence F_n satisfying $\mu_n(F_n)/a \geqq 1$ for every n. The main result is that a weakly convergent sequence of charges contains no eluding load, and from this result several consequences are derived for spaces that are, respectively, locally compact, metric, or compact.

Chapter 6 consists of "addenda." The author's original plan was to devote this chapter to "generalizations and modifications" and to consideration of an invariant measure in locally bicompact groups, but that plan has been abandoned, principally because of the author's publication elsewhere of his work on measure in groups. Chapter 6 is thus confined to application of the concepts of chapters 1–5 to specific examples. *H. Wallman* (Cambridge, Mass.).

Haupt, Otto. Über einen Eindeutigkeitssatz für gewisse Funktionalgleichungen. J. Reine Angew. Math. **186**, 58–64 (1944). [MF 12102]

The author begins by considering a nonempty limit space $\mathfrak{L}$ and "R-systems" (systems of residual sets) S of sets D

in $\mathfrak{L}$; S is an R-system in $\mathfrak{L}$ if the product of a finite or denumerable number of sets of S is also in S, and each set D of S is dense in $\mathfrak{L}$. Let $z=f(x)$ be a single-valued mapping function from a limit space D to a limit space W and let δ and ω be R-systems in D and W, respectively. Suppose that $D'\varepsilon\delta$ and let $W'=f(D')\varepsilon\omega$. The R-systems δ and ω are said to correspond under $z=f(x)$ in (D', W') if the mapping $z=f(x)$ is topological and each subset of D' which is in δ is carried into a set in ω and each subset of W' which is in ω is carried into a set in δ. If $P=L_1\times L_2\times\cdots$ (finite or denumerably many), the L_i being limit spaces, we define the notion of limit in P as follows: $p_0=(p_0^1, p_0^2, \cdots)=\lim_{i\to\infty}p_i$ if and only if $p_0^k=\lim_{i\to\infty}p_i^k$ for each k.

The author then considers equations of the form

$$\Phi[g_0(x, y)]=F\{x, y, \Phi[g_1(x, y)], \Phi[g_2(x, y)], \cdots\},$$

in which $\aleph_0$ terms $g_i(x, y)$ may occur; it is assumed that $x\varepsilon D$, $y\varepsilon D$, $w_i=g_i(x, y)\varepsilon W$, $u_i=\Phi(w_i)\varepsilon U$, D, W, and U all being limit spaces; the functions F and g_i are given and $\Phi(w)$ is to be found. The author proves the following theorem. (1) Let R-systems δ and ω be chosen in D and W, respectively; (2) let $F(x, y, u_1, u_2, \cdots)$ be single-valued and continuous in the product set $D^x\times D^y\times U\times U\times\cdots$, where $D^x\varepsilon\delta$, $D^y\varepsilon\delta$; (3) let $g_i(x, y)$, $i=0, 1, 2, \cdots$ be defined, single-valued and continuous in $D_i^x\times D_i^y$, where $D_i^x\subset D^x$, $D_i^y\subset D^y$, $D_i^x\varepsilon\delta$, $D_i^y\varepsilon\delta$, $i=0, 1, 2, \cdots$; (4) for each fixed $w_0\varepsilon W$, let the equation $w_0=g_0(x, y)$ define a topological map $y=k_0(x; w_0)$ by which a certain set $D_0^x(w_0)\varepsilon\delta$ is mapped into a set $D_0^y(w_0)\varepsilon\delta$ so that δ corresponds to itself under $y=k_0(x; w_0)$ in $(D_0^x(w_0), D_0^y(w_0))$; (5) for each x, let the function $k_0(x; w_0)$ be continuous in w_0 over W; (6) let the maps $w_i=k_i(x; w_0)\equiv g_i[x; k_0(x; w_0)]$ be topological from a set $D_i^x(w_0)\varepsilon\delta$ to a set $W_i(w_0)\varepsilon\omega$, where $D_i^x(w_0)\subset D_0^xD_i^x$, $D_i^y(w_0)\subset D_0^yD_i^y$, and δ and ω correspond under $w_i=k_i(x; w_0)$ in $(D_i^x(w_0), W_i(w_0))$; (7) let $\Phi(w)$ be defined and single-valued in a set $W_0\varepsilon\omega$ with $\Phi(W_0)\subset U$ and suppose that, for all $x\varepsilon D^x$, $y\varepsilon D^y$ for which all the $g_i(x, y)\varepsilon W_0$, $\Phi(w)$ satisfies the functional equation above. Then there is at most one function $\Phi^*(w)$ defined over the whole of W which satisfies the equation and coincides with $\Phi(w)$ on W_0; if $\Phi(w)$ is continuous on W_0, then $\Phi^*(w)$ is continuous on W. The author applies this theorem to obtain results concerning equations of the form

$$\sum_{i=1}^{n} c_i\Phi[g_i(x, y)]=\sum_{j=1}^{p} X_j(x)Y_j(y).$$

C. B. Morrey, Jr. (Aberdeen Proving Ground, Md.).

Krein, Mark, et Krein, Selim. Sur l'espace des fonctions continues définies sur un bicompact de Hausdorff et ses sousespaces semiordonnés. Rec. Math. [Mat. Sbornik] N.S. **13**(55), 1–38 (1943). (French. Russian summary) [MF 11644]

In an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 427–430 (1940); these Rev. **2**, 222] the authors characterized the space $C(Q)$ of all real continuous functions on a bicompact Hausdorff space Q. The characterization, which was made in terms of linear lattice properties, was announced at about the same time by S. Kakutani [Proc. Imp. Acad. Tokyo **16**, 63–67 (1940); these Rev. **2**, 69]. In the present paper the authors solve a more general problem by characterizing those linear subspaces of $C(Q)$ which contain with φ its absolute value. The characterization is given in terms of a normed linear vector lattice which satisfies the condition $|\sup(x, y)|=\max(|x|, |y|)$ for every pair of positive elements x, y as well as the condition $|x|=|x_a|$ (x_a is the absolute value of x). A linear lattice characterization of the spaces $L(0, 1)$, $l^{(\infty)}$, $L(0, 1)\times l^{(\infty)}$ is given. Some of the results proved here have been given by Kakutani [Ann. of Math. (2) **42**, 994–1024 (1941); these Rev. **3**, 205]. *N. Dunford* (New Haven, Conn.).

Šmulian, V. On compact sets in the space of measurable functions. Rec. Math. [Mat. Sbornik] N.S. **15**(57), 343–346 (1944). (Russian. English summary) [MF 12290]

Let μ be a measure defined on T and let A be the class of μ-measurable real-valued functions x on T such that the μ-measure of the set where $|x(t)|>M$ tends to zero as M increases. Then A contains every $L^p(\mu)$, $p>0$, and A contains all μ-measurable functions if $\mu(T)<\infty$. Using an appropriate metric $d(x, x')$ (introduced for Lebesgue measure by E. H. Hanson [Bull. Amer. Math. Soc. **39**, 397–400 (1933)]) such that $d(x_n, x)\to 0$ if and only if x_n converges in μ-measure to x, the author gives a simple proof of a new criterion for compactness (in the sense of convergence in measure) of subsets of A. Analogous conditions are given for compactness of subsets of $L^p(\mu)$, $p>1$. *M. M. Day.*

Augé, Juan. Characterization of a linear functional by the values which it takes on an analytic line. Revista Mat. Hisp.-Amer. (4) **4**, 141–148 (1944). (Spanish) [MF 12168]

If F is a linear functional defined over analytic functions (analytic lines) $y(t)$, the character of F is completely determined by its value on the function $(\alpha-t)^{-1}$, where α varies over the exterior of the domain of $y(t)$. The author determines conditions on a function $z(t, \alpha)$ in order that the behavior of F on $z(t, \alpha)$ determine the value of F for any $y(t)$. *E. R. Lorch* (New York, N. Y.).

Hille, Einar. Remarks on ergodic theorems. Trans. Amer. Math. Soc. **57**, 246–269 (1945). [MF 12133]

Let T be a bounded linear transformation of a Banach space to itself. The sequence of powers T^n does not ordinarily converge but may be summable to a projection. That is, the series $I+\sum_{n=1}^{\infty}(T^n-T^{n-1})\lambda^{-n}$ may be summable by some method for $\lambda=1$. When the series converges its sum is $(\lambda-1)R(\lambda)$, where $R(\lambda)$ is the resolvent of T. Classical ergodic theorems assert the $(C, 1)$ summability of this series, either in the strong or the a.e. topology of transformations. In the present paper it is shown, in analogy with Abelian and Tauberian theorems for numerically valued series, that (C, k) summability implies Abel summability, and that Abel summability plus $\|T^n\|\leqq M$ implies (C, k) summability for every $k>0$. A detailed study is made of the transformations

$$T_\alpha f=f(t)-(1/\Gamma(\alpha))\int_0^t(t-u)^{\alpha-1}f(u)du$$

on the spaces $C_0[0, 1]$ and $L_1(0, 1)$. Here the resolvent has an essential singularity at $\lambda=1$ and no other singularities. In the strong topologies in the two spaces, the Abel limit exists for $0<\alpha<2$, the (C, k) limit for $\alpha=1$ and $k>\frac{1}{2}$, and $\lim T_\alpha^n$ for $0<\alpha<1$, and similarly in the a.e. topology in L_1. These results bear on the question, studied by Dunford [same Trans. **54**, 185–217 (1943); these Rev. **5**, 39], as to the character of $\lambda=1$ for the resolvent when an ergodic theorem holds for T. *J. C. Oxtoby* (Bryn Mawr, Pa.).

TOPOLOGY

Barbălat, B. Sur les espaces topologiques les plus généraux. Bull. Math. Soc. Roumaine Sci. **45**, 103–112 (1943). [MF 12761]

The author calls a set $E=\{x, y, \cdots\}$ a space (T) [not the same as a "T-space" in the terminology of Alexandroff and Hopf] if an operation $(')$ is defined with the property that for any $M \subset E$, $x \varepsilon M'$ if and only if $M-x$ is not void and $x \varepsilon (M-x)'$. Then M is closed if $M' \subset M$; M is open if it coincides with the complement of $\sum X'$, $X \subset N$, where N is the complement of M. The proposition that the complement of an open set is a closed set is true for spaces (T); but examples show that the converse may fail for simple spaces (T). The paper is devoted to an investigation of conditions which insure that the complement of a closed subset of a space (T) is open; for example: $M \subset S$ implies $M' \subset S'$. *R. Arens* (Princeton, N. J.).

Freudenthal, Hans. Über die Enden diskreter Räume und Gruppen. Comment. Math. Helv. **17**, 1–38 (1945).

The theory of compactifying spaces by adding new points called ends has been studied by Freudenthal in previous papers [Math. Z. **33**, 692–713 (1931); Ann. of Math. (2) **43**, 261–279 (1942); these Rev. **3**, 315]. Contributions have also been made by Zippin [Amer. J. Math. **57**, 327–341 (1935)]. H. Hopf has studied ends of spaces on which there is defined a group of transformations [Comment. Math. Helv. **16**, 81–100 (1943); these Rev. **5**, 272]. The present paper carries out an investigation of ends for discrete spaces and for abstract groups. A discrete space, as the term is used here, is a set of elements in which certain pairs are designated as neighbors, this relation being symmetric and reflexive. A polygon, for example, leads to a discrete space if its vertices are called points, and provided two points are called neighbors if they are identical or end points of the same segment. If a space R is discrete and satisfies certain conditions it is shown that, by adding certain points to R, a space R^* can be formed whose only nonisolated points are the end points. The space R^* is zero-dimensional and compact.

Let G be a group with a finite number of generators. Then G is made into a discrete space in the following way. Let U be a subset of G with the properties (1) U is finite, (2) U includes the identity, (3) U includes the inverse of each of its elements, (4) U generates G. Then a and b are called neighbors if $a^{-1}b$ is in U. This enables one to utilize the theory of ends for discrete spaces. The particular choice of U is of no consequence. The set of end points consists of 0, 1, or 2 points, or is perfect, and only finite groups have 0 end points. There are 0, 1, or 2 end points which are left invariant; groups are called elliptic, parabolic, or hyperbolic according to which of these possibilities occurs. These concepts are developed in various directions.

D. Montgomery (Princeton, N. J.).

Eckmann, Beno. L'idée de dimension. Revue de Théologie et de Philosophie (N.S.) **31** (no. 127), 65–79 (1943). [MF 12416]

Eckmann, Beno. The idea of dimension. Gaz. Mat., Lisboa **4**, no. 17, 4–6 (1943); **5**, no. 18, 4–8 (1944). (Portuguese) [MF 12420]

The first paper is an expository lecture; the second is a translation of the first.

Albuquerque, J. The theorems of Netto and Lüroth and the concept of dimension. Gaz. Mat., Lisboa **5**, no. 19, 1–4 (1944). [MF 12418]

The theorems in question are that a square is not homeomorphic to a line-segment, and that a plane is not homeomorphic to a line.

Weinstein, I., and Kajdan, J. Finite-multiple continuous dimension-raising mappings. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 129–138 (1944). (Russian. English summary) [MF 11463]

The paper is concerned with a continuous closed mapping f of a space X onto a space Y under which the dimension is increased: $\dim Y > \dim X$. By a kernel of a point $y \varepsilon Y$ is meant any set $A \varepsilon f^{-1}(y)$ satisfying the following conditions. (1) The image of every neighborhood $U(A)$ contains y as inner point. (2) There is no proper subset $A_0 \subset A$ satisfying (1). It is easy to see that every point of Y has at least one kernel.

Suppose that f is $(n+1)$-multiple and $\dim Y - \dim X = n$. According to a result of the reviewer [J. Reine Angew. Math. **169**, 71–78 (1933)], for every $k=1, 2, \cdots, n+1$ there exists at least one point of multiplicity k. The author generalizes this result by showing that there exists a point of multiplicity k such that its unique kernel coincides with $f^{-1}(y)$. Along similar lines the author establishes several other results, in particular the following theorem. If f is a finite-multiple mapping of X on Y and $\dim Y - \dim X = n$, then there exist n points $x_1, \cdots, x_n$ with coinciding images $f(x_1) = \cdots = f(x_n)$ and an open set U containing all $x_1, \cdots, x_n$ such that y is not an inner point of $f(U)$.

W. Hurewicz (Cambridge, Mass.).

Aumann, Georg. Über Räume mit Mittelbildungen. Math. Ann. **119**, 210–215 (1944). [MF 11896]

A space R is an m_n-space if there is a point function $m(x_1, \cdots, x_n)$ corresponding to each ordered ennuple of points of R such that (a) $m(x_1, \cdots, x_n)$ is continuous and symmetric in $x_1, \cdots, x_n$; (b) $m(x, \cdots, x) = x$ for every x in R. A function $m(x_1, \cdots, x_n)$ is called a mean. The property of being an m_n-space is topologically invariant. An ingenious proof shows that there can be no mean on a circle. The same theorem is true for a k-sphere $(k \geqq 1)$. If R is an m_n-space, so is any "Retraktteilraum," that is, any subspace R' such that there is a continuous map of R on R' which is the identity on R'. If R is an m_n-space, so is any component of R, that is, any maximal connected subset. Among all connected one-dimensional complexes, the trees are the only m-spaces; and any tree is an m_n-space for every $n \geqq 2$. Among all two-dimensional manifolds (closed and open surfaces) the open circular disc is the only m-space.

A. B. Brown (Flushing, N. Y.).

Young, Gail S., Jr. Spaces in which every arc has two sides. Ann. of Math. (2) **46**, 182–193 (1945). [MF 12396]

This paper considers a connected, locally connected, complete Moore space S which is subjected to one or more of the following additional restrictions. (a) If AB is a closed arc, and D is a domain containing the open arc (AB), then there is a domain E, lying in D and containing (AB), which is separated by this open arc into two connected domains, each of which contains AB in its boundary. (b) If AB is a

closed arc and D is a domain containing (AB), then D contains a connected domain E, containing (AB), which is separated by this open arc in such a way that AB lies in the boundary of every component of $E-(AB)$. (c) If P is a point and R is a region containing P, there is a domain D, containing P and lying in R, such that, if APB is an arc in D, then D contains two connected domains bordering (APB) in D. Two connected subdomains E and F of a domain D are said to border the arc segment (AB) in D provided (AB) is contained in D, $E+(AB)+F$ is a domain, and AB lies in the closures of both E and F.

It is shown that, in the space S, (a) is a consequence of either (b) or (c). If S satisfies (a) then: (1) each point of S is contained in a connected domain which is separated by every simple closed curve lying in it; (2) it is not necessarily true that every two points of S can be separated by a simple closed curve; (3) if S is locally compact it is locally the plane; if it is locally compact and metric, it is a 2-manifold without boundary, which is open if S is not compact, and closed if S is compact; (4) the space S locally satisfies axioms 0–5 of R. L. Moore.

F. B. Jones [Trans. Amer. Math. Soc. **42**, 53–93 (1937)] introduced the following axiom. If P is a point and R is a region containing P, then there exists in R a connected domain D containing P such that the boundary of D is the sum of a finite number of continua. The author shows that, if S is metric and satisfies (a) and this axiom, then S can be imbedded in a 2-manifold without boundary. A characterization of 2-manifolds with boundary is also given along similar lines. *D. W. Hall* (College Park, Md.).

Wallace, A. D. A fixed-point theorem. Bull. Amer. Math. Soc. **51**, 413–416 (1945). [MF 12521]

The principal result is the following theorem. If S is a compact connected Hausdorff space, $TS=S$ a topological transformation, and $e=Te$ an end point, then (a) there is a continuum $H \subset S-e$, having no cut point, with $TH=H$; (b) if S is locally connected, there is a fixed point in $S-e$. The argument is based on what is essentially an order relation. A rather general theorem of this type is proved for a partially ordered space. Part (b) of the principal result is of particular interest. If T is fixed-point-free, it gives some structural information about any invariant closed set; if T has fixed points the same holds for the set of fixed points. *J. L. Kelley* (Aberdeen Proving Ground, Md.).

Youngs, J. W. T. The topological theory of Fréchet surfaces. Ann. of Math. (2) **45**, 753–785 (1944). [MF 11381]

The author begins by defining a Fréchet surface as a class of equivalent mappings, the equivalence being that of Fréchet, also defined by the author. Let $Y=f(X)$ be a continuous mapping of a 2-sphere S into a subset of some metric space. By the Eilenberg-Whyburn factor theorem, there are mappings $Z=m(X)$ of S onto a set Σ and $Y=l(Z)$ such that $f(X)\equiv l[m(X)]$, $Z=m(X)$ is monotone and $Y=l(Z)$ is light. From a well-known theorem of R. L. Moore, Σ is a cactoid, that is, a Peano space each of whose nondegenerate cyclic elements is a 2-sphere. Suppose that σ is a 2-sphere of Σ. The author defines the mapping $Z=m(\sigma, X)$ as the mapping m followed by the monotone retraction of Σ onto σ; $Z=m(\sigma, X)$ is a monotone map of S onto σ and has degree $+1$ or -1 if S and σ are oriented.

The principal motivation of this paper is to develop a topological theory of Fréchet surfaces to replace the erroneous one developed by the reviewer [Amer. J. Math. **57**, 17–50 (1935)] and to carry through the applications made by the reviewer to the problem of the area of surfaces [Amer. J. Math. **58**, 313–322 (1936)]. The erroneous theorem stated by the reviewer [and also by Kerékjártó, Acta Litt. Sci. Szeged. **3**, 49–67 (1927)] is equivalent to the following. Let $Y=f_i(X_i)$, $X_i \in S_i$, be two mappings as above with factorizations $Z_i=m_i(X_i)$, $Y=l_i(Z_i)$ and cactoids Σ_i as above. Then a necessary and sufficient condition that the two maps be Fréchet equivalent is that there exist a homeomorphism $Z_2=h(Z_1)$ such that $l_2[h(Z_1)]\equiv l_1(Z_1)$ on Σ_1. The present author's principal result is the correct theorem which is the same as the above, except that it is necessary and sufficient that the homeomorphism $Z_2=h(Z_1)$ be such that the degree of the partial mapping between any two corresponding 2-spheres σ_i of Σ_i be $+1$ for every pair, or -1 for every pair, S_1, S_2, the σ_i all being oriented so that the degrees of the mappings $m_i(\sigma_i, X_i)$ are all $+1$.

Let A be a Fréchet surface defined by the mapping $Y=f(X)$, $X \in S$, and let $Z=m(X)$ and $Y=l(Z)$ be a factorization of $Y=f(X)$ and let $\Sigma=m(S)$. Let $\sigma_1, \sigma_2, \cdots$ be the 2-spheres of Σ and let A_i be the Fréchet surfaces defined by the mappings $Y=l[m(\sigma_i, X)]$. The author concludes the paper with a proof of the reviewer's result (based on the erroneous theorem) that the Lebesgue area of A is the sum of those of the A_i. *C. B. Morrey, Jr.*

Shanin, N. On imbedding in a power of topological space. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 233–242 (1944). (Russian. English summary) [MF 12300]

Let S and R be topological spaces and f a (continuous) mapping of S onto R. A set $G \subset S$ is said to be realized by f if there is an $H \subset R$ such that $G=f^{-1}(H)$. The author is interested in the problem of when a family Φ of mappings f realizes all the sets of an open subbase for the space S. If such a family Φ exists, S is said to be topologically subjected to R. Several theorems of the following type are proved. A T_0-space S can be topologically imbedded into the product space R^{Ξ}, where R is a T-space and Ξ is a cardinal number, if and only if there is a family Φ of power at most Ξ of mappings $S \to R$ which realizes an open subbase for S. *S. Eilenberg* (New York, N. Y.).

Fox, Ralph H. On topologies for function spaces. Bull. Amer. Math. Soc. **51**, 429–432 (1945). [MF 12523]

The problem treated is that of the relation between the concepts "homotopy of a mapping" and "path in function space." Let X, Y, T be three topological spaces; with every mapping h of $X \times T$ into Y, continuous in x for fixed t, is associated in a natural way a mapping h^* of T into the set F of all continuous maps of X into Y, and conversely. (The most interesting T is, of course, the unit interval.) The set F can be topologized; in particular, the compact-open topology is obtained by taking as a subbasis for open sets the set $M(A, W)$ of those mappings which map a compact subset A of X into an open subset W of Y. It is shown that, with the compact-open topology for F, continuity of h is equivalent to continuity of h^*, (1) for arbitrary T if X is regular and locally compact, (2) if X and T satisfy the first countability axiom. The local compactness in (1) is important as is shown by the following result. If X is separable metric and Y is the real line, then it is possible so to topologize F that continuity of h and h^* are equivalent for any T, if and only if X is locally compact. *H. Samelson.*

Fox, Ralph H. Torus homotopy groups. Proc. Nat. Acad. Sci. U. S. A. **31**, 71–74 (1945). [MF 11973]

By an appropriate definition of the r-dimensional torus homotopy groups, results of J. H. C. Whitehead may be formulated thus: "Let K be a complex obtained from a complex K^* by removing the interior $\sigma-\dot{\sigma}$ of a principal n-dimensional simplex σ, where $n>2$. The nucleus of the injection homomorphism $\tau_n(K)\to\tau_n(K^*)$ is precisely the invariant subgroup of $\tau_n(K)$ which is generated by the image of the injection homomorphism $\tau_n(\dot{\sigma})\to\tau_n(K)$." This suggests a method of attack on the harder problem where $\dim\sigma>n$. *H. M. Gehman* (Buffalo, N. Y.).

Whitehead, J. H. C. On the groups $\pi_r(V_{n,m})$ and sphere-bundles. Proc. London Math. Soc. (2) **48**, 243–291 (1944). [MF 11385]

The rectangular coordinate systems of the m-dimensional Euclidean subspaces of Euclidean n-space which have their origin at a point O constitute the points of a $\frac{1}{2}m(2n-m-1)$ dimensional manifold $V_{n,m}$. The r-dimensional homotopy groups $\pi_r(V_{n,m})$ of $V_{n,m}$ were calculated by Stiefel [Comment. Math. Helv. **8**, 305–353 (1936)] for $r\leqq k=n-m$. The main object of this paper is to calculate $\pi_r(V_{n,m})$ for $r=k+1$ and $r=k+2$. The results are rather complicated and present no perceptible regularity.

The principal tool is a generalization of Freudenthal's theorems [Compositio Math. **5**, 299–314 (1938)] on the "Einhängung"; essentially the generalization consists of replacing the southern hemisphere of the "Einhängung" by an arbitrary complex. Slightly simplified, the main result [theorem 8] of this section is as follows. Let X^* be a complex, let E^t be a principal t-dimensional simplex and let X be obtained from X^* by removing the interior of E^t. Denote by h_r and ϕ_r the natural homomorphisms $\pi_r(\dot{E}^t)\to\pi_r(X)$ and $\pi_r(X)\to\pi_r(X^*)$. If $r<2t-2$ and $\pi_s(X)$ is trivial for $1\leqq s\leqq r-t+2$ then the image of h_r is the nucleus of π_r and the nucleus of h_{r-1} is isomorphic to $\pi_r(X^*)-\phi_r(\pi_r(X))$. Weaker and stronger conclusions can be drawn if the hypotheses are weakened and strengthened.

In the main, the procedure for calculating $\pi_r(V_{n,m})$ is as follows. Choose a point $(a_1,\cdots,a_m)$ in $V_{n,m}$. Reflection of $(a_1,\cdots,a_m)$ about any hyperplane through O determines a point $(v_1,\cdots,v_m)$ in $V_{n,m}$. The set of points $(v_1,\cdots,v_m)$ obtained by reflecting $(a_1,\cdots,a_m)$ through those hyperplanes which contain $a_1,\cdots,a_{n-r-1}$ (where $0\leqq n-r-1<m$) is a subset of $V_{n,m}$ which is denoted by P^r_{n-1}. Since the hyperplanes involved constitute a projective r-space P^r there is defined a fundamental mapping ϕ of P^r onto P^r_{k-1}. (This mapping can be characterized as a shrinking of the subset P^{k-1} of P^r to a point.) A rather complicated "singular reticulation" of $V_{n,m}$ is described, the $(r+1)$-dimensional skeleton of which is P^l_{k-1}, where $l=\min(r+1,n-1)$, if $r<2k$. Thus the calculation of $\pi_r(V_{n,m})$ is reduced in most cases to the calculation of $\pi_r(P^l_{k-1})$. The P^r_{k-1} form an expanding sequence of subsets; $P^r_{k-1}-P^{r-1}_{k-1}$ is an open r-cell, and P^r_{k-1} is obtained from P^{r-1}_{k-1} and a closed r-cell E^r by identifying points of the boundary $\dot{E}^r=S^{r-1}$ with their images under the mapping ϕu (where u denotes the usual covering mapping $S^{r-1}\to P^{r-1}$). With this framework, $\pi_r(V_{n,m})$ is calculated by applying theorem 8 with $t=l$ or by applying similar theorems of the author [Proc. London Math. Soc. (2) **45**, 243–327 (1939), in particular, theorem 18; Ann. of Math. (2) **42**, 409–428 (1941), in particular, lemma 4; these Rev. **2**, 323]. Use of this method to calculate $\pi_r(V_{n,m})$ for $r\geqq k+3$ would appear to require more knowledge of the homotopy groups of spheres than is at present available, as well as more general theorems of the type of theorem 8.

The remainder of the paper is devoted to various auxiliary theorems. The homomorphic relationships between the homotopy groups of a space and of a subset are studied (also of a fibre space, base space and fibre); many of the results have been obtained independently by other authors [Borsuk, Hurewicz, Steenrod, Alexandroff, G. W. Whitehead and the reviewer]. The order of the element of $\pi_r(P^r_{k-1})$ determined by the map ϕu is calculated for all values of r and k. The author proves that, if ψ_1 and ψ_2 are mappings of P^r into a space X such that $\psi_1|P^{r-1}$ and $\psi_2|P^{r-1}$ are homotopic, then $\psi_1 u$ and $\psi_2 u$ determine elements of $\pi_r(X)$ which are congruent mod 2; moreover, if ψ is a mapping of P^r into S^{r-1} then ψu is (a) inessential if $r\not\equiv 3 \pmod 4$, and (b) essential if $r\equiv 3 \pmod 4$ and $\psi|P^{r-1}$ is essential. In an appendix the author shows that the homotopy groups of a nondegenerate quadric in projective space of $n-1$ complex variables can be calculated in terms of $\pi_r(V_{n,2})$.

R. H. Fox (Syracuse, N. Y.).

Kiang, Tsai-Han. The manifolds of linear elements of an n-sphere. Bull. Amer. Math. Soc. **51**, 417–428 (1945). [MF 12522]

The author studies the topology of the manifolds of oriented and nonoriented linear elements of an n-sphere ($n>1$) and determines, in particular, their fundamental groups and Betti groups by a direct method. For the manifold of oriented linear elements the nonvanishing (integral) Betti groups are, for even n, of dimensions 0, $n-1$, $2n-1$; for odd n, of dimensions 0, $n-1$, n, $2n-1$. All these groups are free cyclic, except the one of dimension $n-1$ for even n, which is cyclic of order two. Analogous but more complicated results are found for the manifold of nonoriented linear elements of the n-sphere. *S. Chern.*

Eilenberg, Samuel, and Steenrod, Norman E. Axiomatic approach to homology theory. Proc. Nat. Acad. Sci. U. S. A. **31**, 117–120 (1945). [MF 12321]

The authors present in outline an axiomatic treatment of homology theory and suggest, with some plausibility, that, at least from the pedagogical point of view, their approach is superior to the usual one based on complexes. The primitive concepts are: Abelian relative homology groups $H_q(X,A)$, where X is an arbitrary topological space and A any closed subset of X (write $H_q(X)$ when A is empty); a boundary operator $\partial: H_{q+1}(X,A)\to H_q(A)$; an induced homomorphism $f_*: H_q(X,A)\to H_q(Y,B)$ associated with each mapping $f: X\to Y$ such that $f(A)\subset B$. Among the axioms, of which there are seven, are the relations $(gf)_*=g_*f_*$, $\partial f_*=f_*\partial$; the invariance of f_* under homotopy of f; and the assumption that, if P is a point, $H_q(P)=0$ when $q\geqq 1$. Then $H_0(P)$ is defined to be the coefficient group of the theory. For complexes X the whole of homology theory in the usual sense can be derived from the axioms, the groups corresponding to subcomplexes A being unique when the coefficient group is specified. The Čech theory satisfies the axioms, as does the singular theory. *P. A. Smith* (New York, N. Y.).

Hopf, Heinz. Über die Bettischen Gruppen, die zu einer beliebigen Gruppe gehören. Comment. Math. Helv. **17**, 39–79 (1945).

Previous papers [same Comment. **14**, 257–309 (1942); **15**, 27–32 (1942); these Rev. **3**, 316; **4**, 173] dealt with the

influence of the fundamental group of a complex upon the higher homology groups. The present paper is a far reaching generalization of the previous results. For any discrete group G and any ring J with a unit, Abelian "homology groups" $G^n{}_J$ are defined for $n=0, 1, \cdots$. The definition involves several choices of generators in various groups but is ultimately shown to be independent of the choices made. Let K be a simplicial complex and G a group of simplicial transformations of K into itself without fixed points; K may then be regarded as a regular covering of a complex $\mathfrak{R}$ with G as the group of covering transformations. For a ring J (with a unit) as coefficient group, the homology groups $H^n(K, J)$ and $H^n(\mathfrak{R}, J)$ are considered and a homomorphism $U: H^n(K, J) \to H^n(\mathfrak{R}, J)$ is defined using the projection $U: K \to \mathfrak{R}$. Assuming that $H^n(K, J)=0$ for $n=0, 1, \cdots, N$, it is shown that $H^n(\mathfrak{R}, J) \cong G^n{}_J$ for $n=0, 1, \cdots, N-1$ and $H^N(K, J)/UH^N(K, J) \cong G^N{}_J$.

If K is taken to be the universal covering of $\mathfrak{R}$ the theorem implies that if the homotopy groups $\pi_n(\mathfrak{R})$ vanish for $0<n<N$ then the homology groups $H^n(\mathfrak{R}, J)$ for $0<n<N$ and the group $H^N(\mathfrak{R}, J)/S^N(\mathfrak{R}, J)$ are determined by J and the fundamental group $\pi_1(\mathfrak{R})$ (here $S^N(\mathfrak{R}, J)$ denotes the subgroup of $H^N(\mathfrak{R}, J)$ generated by the spherical cycles). This particular theorem is contained in a theorem announced by S. Eilenberg and S. MacLane [Proc. Nat. Acad. Sci. U. S. A. 29, 155–158 (1943); these Rev. 4, 224].

Using these topological theorems the groups $G^n{}_J$ are computed for various G and J. For special classes of complexes K (for instance, Poincaré spheres) limitations are obtained on the groups G that can act on K (without fixed points).

S. Eilenberg (New York, N. Y.).

Mayer, Walther. The duality theory and the basic isomorphisms of group systems and nets and co-nets of group systems. Ann. of Math. (2) 46, 1–28 (1945). [MF 11786]

The author defines in the following manner the homology groups with respect to an Abelian group k. Given the group system Σ, consisting of chain groups $L_i=L_i(\Sigma)$ $(-\infty<i<\infty)$ and boundary operator $R: L_{i+1} \to L_i$, with $RR=0$, the k character system Σ^k has chain groups $L_i(\Sigma^k)=(L_{-i}(\Sigma))^k$ (that is, the group of homomorphisms of $L_{-i}(\Sigma)$ into k), where, for $f \in L_i(\Sigma^k)$, Rf is defined to be the homomorphism fR of $L_{-i+1}(\Sigma)$ into k. Let $Z_i(\Sigma) \subset L_i(\Sigma)$ be the i-dimensional cycle group of Σ and K_i be the annihilator in $L_i(\Sigma)$ of $Z_{-i}(\Sigma^k)$. (K_i contains but is not in general equal to the closure of the group of bounding cycles in $L_i(\Sigma)$.) The i-dimensional homology group $D_i(\Sigma)$ with respect to k is defined to be Z_i-k_i. There is a natural multiplication of $D_i(\Sigma)$ and the analogously defined $D_{-i}(\Sigma^k)$ into k, under which no element of either group annihilates the other group. However, $D_i(\Sigma)$ is not determined by $D_{-i}(\Sigma^k)$ and k. The author shows that, if Σ is the system obtained from a simplicial complex K with coefficient group G, the groups D_i are topological invariants of $|K|$.

In case each chain group of Σ is locally compact and k is the group of the reals mod 1, $D_i(\Sigma)$ is identical with the ordinary homology group $H_i(\Sigma)=Z_i-$[closure of group of bounding cycles]. The author investigates, for the case where all chain groups of Σ are either compact or discrete and k is the group of reals mod 1, the relations between a group system Σ, a subsystem M and the system of residue class groups $\Sigma-M$. Among the results are the following. (1) If (Σ^k, M) is the system of annihilators of groups of M, then $B_i(\Sigma^k, M)$ is dual to $B_{-i}(\Sigma-M)$. (If M consists of the zero elements of the groups of Σ this is the Poincaré-Pontrjagin duality theorem for group systems.) (2) Alexander-Pontrjagin theorem for group systems: if $B_i(M, \Sigma)$ denotes the subgroup of $B_i(M)$ consisting of classes of cycles bounding in Σ, then $B_i(M, \Sigma)$ is dual to $B_{-i-1}[(\Sigma^k, M), \Sigma^k]$.

The author extends the theory, with modifications, to nets and co-nets [see Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloquium Publ., v. 27, New York, 1942, in particular, chap. 6; these Rev. 4, 84]. As an application of the theory, relations of the classical type involving the homology groups of the union and intersection of two systems are derived and extended to nets. *J. L. Kelley.*

Mayer, Walther. On products in topology. Ann. of Math. (2) 46, 29–57 (1945). [MF 11787]

The author discusses Whitney type cup and cap products in some detail. A uniqueness theorem is given, holding in an algebraic structure somewhat more general than a complex. The cup product is then defined explicitly for a simplicial complex, as well as a geometric variant of this product, which is, however, homologically the same. An intersection of singular cycle classes is defined for an oriented manifold.

J. L. Kelley (Aberdeen Proving Ground, Md.).

Tietze, Heinrich. Über Simony-Knoten und Simony-Ketten mit vorgeschriebenen singulären Primzahlen für die Figur und für ihr Spiegelbild. Math. Z. 49, 351–369 (1944). [MF 12000]

The author studies torus knots and linkages of types $[2, 2n-1]$ and $[2, 2n]$, called (special) Simony-figures after their discoverer. A singular prime number of a Simony-figure L is a prime for which the Minkowski unit of the quadratic form of L [Reidemeister, Knotentheorie, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. I, no. 1, Springer, Berlin, 1932] has the value -1. The author shows that for two given sets of primes (satisfying a trivial condition) there always exist an infinite number of Simony figures such that the two sets are the sets of singular primes for the figure and its reflection, respectively; a condition is given that among the figures so obtained there are knots (and not only linkages). *H. Samelson* (Syracuse, N. Y.).

Tietze, Heinrich. Über spezielle Simony-Knoten und Simony-Ketten mit vorgeschriebenen singulären Primzahlen. II. Monatsh. Math. Phys. 51, 85–100 (1944). [MF 12478]

[Part I appeared in the same Monatsh. 51, 1–14 (1943).] The knots and linkages mentioned in the title are torus knots and linkages of type $[2, n]$. The author determines, for an arbitrary set $\mathfrak{P}$ of odd primes, all numbers n such that the torus knot or linkage $[2, n]$ has exactly the elements of $\mathfrak{P}$ as singular primes (singular in the sense of the quadratic form of a knot). The problem consists in finding the solutions of a number of equations with quadratic residue symbols. *H. Samelson* (Syracuse, N. Y.).

Dukor, I. G. On a theorem of Helly on collections of convex bodies with common points. Uspekhi Matem. Nauk 10, 60–61 (1944). (Russian) [MF 12672]

A theorem of E. Helly [Jber. Deutsch. Math. Verein. 32, 175–176 (1923)] states that, if a collection of at least $n+1$ convex bodies, not all of them extending to infinity, in

Euclidean n-space is such that an arbitrary $n+1$ of them have at least one point in common, then all of them have at least one point in common. The author gives a new proof by showing that, if an arbitrary m of them intersect ($m \geqq n+1$), then any $m+1$ of them intersect.

E. F. Beckenbach (Los Angeles, Calif.).

Lannér, Folke. On convex bodies with at least one point in common. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **13**, no. 5, 41–50 (1943). [MF 12590]

Let $K_1, \cdots, K_N$ be a set of convex bodies in n-dimensional Euclidean space R_n; $H_1(u), \cdots, H_N(u)$, their supporting functions; $u_1, \cdots, u_N$, a set of arbitrary vectors. The following theorem is proved. A necessary and sufficient condition that $K_1, \cdots, K_N$ have at least one point in common is that $\sum_{i=1}^{N} H_i(u_i) \geqq 0$ for all u_i satisfying $\sum_{i=1}^{N} u_i = 0$. If the condition is satisfied and if all K_i possess inner points, the equality sign can be attained for a suitable set of u_i, not all of which vanish, if and only if the intersection of a number of K_i is identical with the intersection of their boundary points. The sufficiency proof is carried out first for K_i which contain inner points and have unique supporting planes. The general case then follows through approximation of the given set by a sequence of the special type followed by a passage to the limit. The author makes use of his theorem to give a new proof of the following theorem of E. Helly. A set of convex bodies in R_n have a common point if and only if any $n+1$ of them have a common point.

J. J. Stoker (New York, N. Y.).

Brooks, R. L. On colouring the nodes of a network. Proc. Cambridge Philos. Soc. **37**, 194–197 (1941). [MF 12243]

The author proves that, if $n > 2$, the nodes of a network or linear graph can be colored in n colors if the following two conditions hold. (1) At most n lines meet at each node. (2) The network contains no n-simplex. *P. Franklin.*

Hadwiger, H. Über eine Klassifikation der Streckenkomplexe. Vierteljschr. Naturforsch. Ges. Zürich **88**, 133–142 (1943). [MF 12239]

A network is called a $K(k)$ if by a certain contraction process one may obtain from it a $(k-1)$-simplex but not a k-simplex. The author proves that, for $k \leqq 3$, the nodes of a $K(k)$ may be colored in k colors. He also derives an inequality involving k, the number of lines, and the number of points. *P. Franklin* (Cambridge, Mass.).

Blanusa, Danilo. Über die Anzahl der Bedingungsgleichungen beliebigen geodätischen Netzen. Z. Vermessungswesen **73**, 54–62 (1944). [MF 12622]

This is an application to surveying of the formula $P = \alpha_1 - \alpha_0 + s$ for the connectivity of a graph with α_1 branches and α_0 nodes, consisting of s separate pieces. [See K. Reidemeister, Einführung in die Kombinatorische Topologie, Vieweg, Braunschweig, 1932, p. 106, or D. König, Theorie der Endlichen und Unendlichen Graphen, Akademische Verlagsgesellschaft, Leipzig, 1936, pp. 53–57.] The relative positions of several points are determined by observing the directions of some of them as seen from others. Two points are joined by a "one-sided" or "two-sided" line-segment according as one point is observed from the other or both points are observed from each other. Topological considerations are applied to the graph formed by just the "two-sided" segments (with their end points). The most economical set of observations will be such as will make this graph a tree. The connectivity of the graph is what the author calls "the number of independent polygon-equations." *H. S. M. Coxeter* (Toronto, Ont.).

Ríos, Sixto. The problem of the number of isomers in the homologous series of organic chemistry. Investigación y Progreso. Madrid **15**, 347–354 = Gaz. Mat., Lisboa **6**, no. 25, 1–4 (1944). (Spanish) [MF 12791]

An exposition of the results of Pólya [Acta Math. **68**, 145–254 (1937)].

MATHEMATICAL PHYSICS

Deans, J. The mathematical theory of the influence of thin films on the reflection and transmission of light. Math. Gaz. **29**, 57–65 (1945). [MF 12536]

Mathematical expressions for the intensities of the reflected and transmitted beams of light for all angles of incidence of a plane polarized plane wave are determined for the case of two transparent media separated by a thin film with parallel plane surfaces. The method consists of summation of the components of the electric and magnetic intensities of the waves forming the resultant reflected and transmitted waves, a process which yields the amplitudes as well as the phase differences. The reflected and transmitted waves are shown to be, in general, elliptically polarized. The variation of the amplitudes with the thickness of the films is discussed and the maxima and minima of these functions are determined for different values of the index of the film. Finally, the efficiency of the film with white light is discussed. *P. Boeder* (Southbridge, Mass.).

Blažko, S. N. On the astronomical refraction. Astr. J. Soviet Union [Astr. Zhurnal] **21**, 307–318 (1944). (Russian. English summary) [MF 12507]

The present theory of refraction differs from other theories known to me in that: (1) the relation between the coefficient of refraction of air μ and its density δ is assumed $\mu - 1 = c\delta$ instead of the usual $\mu^2 - 1 = c\delta$, it being more exact physically and leading to simpler mathematical treatment; (2) numerous changes of the independent variable are avoided; (3) the expression of refraction is developed in a series of the form $\operatorname{tg}\zeta \sec^{2n}\zeta$, $n = 0, 1, 2, \cdots$, whereby the coefficients become polynomials instead of infinite series.

From the author's summary.

Sobolev, V. On the approximate solution of a problem of scattering of light in a medium with an arbitrary scattering diagram. Astr. J. Soviet Union [Astr. Zhurnal] **20**, 14–22 (1943). (Russian. English summary) [MF 12393]

A solution of the problem of the scattering of light in a medium with any scattering diagram is given. The problem is set up for a scattering medium bounded by two parallel planes and is reduced to the solution of a differential and an integro-differential equation. The scattering of the first order is taken into account approximately by keeping only the first two terms in the development in a series of Legendre polynomials of the function defining the scattering diagram. The main results of the paper are given by two formulas which give the factor of brightness. The special case when the medium is bounded by a bottom with a given albedo is examined. Some applications of the results to the terrestrial atmosphere are pointed out. The paper concludes with some

critical remarks concerning the method of L. V. King [Philos. Trans. Roy. Soc. London. Ser. A. 212, 375–433 (1913)]. *H. P. Thielman* (Ames, Iowa).

Fursov, V., Belenjkij, S., and Galanin, A. Variation of density and scattering of light in Bose-Einstein and Fermi-Dirac gases. Uchenye Zapiski Moskov. Gos. Univ. Fizika **74**, 59–66 (1944). (Russian) [MF 12718]

The intensity of scattering of light in Bose-Einstein and Fermi-Dirac gases is calculated using the idea of fluctuations of density. A general formula is obtained, reducing to the classical one for high temperatures but giving a generally distinct result for low temperatures. In particular, in a Bose-Einstein gas scattering of light is strongly increased with approach to the condensation point; this suggests an experimental possibility for investigating the hypothesis of the connection between the anomalous behavior of liquid helium II and the properties of a degenerate Bose gas.

From the authors' summary.

Cerrillo, Manuel. Simple method of establishing the canonical equations of electrical networks. Comisión Impulsora y Coordinadora de la Investigación Científica. (Mexico). Anuario **1943**, 179–184 (1944). (Spanish) [MF 12347]

***Carter, G. W. The Simple Calculation of Electrical Transients.** Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1945. viii+120 pp. $1.75.

This monograph on the applications of the Heaviside calculus to problems of linear lumped circuits is addressed primarily to engineers whose mathematical background is not very great. The book opens with a comparison of the steady state technique of Steinmetz with the transient technique of Heaviside. The following topics are discussed: (a) transient solutions in linear circuits due to constant impressed electromotive forces, (b) transient solutions in such circuits due to arbitrary impressed electromotive forces, (c) impedance and admittance operators and finally (d) resonance, damping and stability. Many examples from practice are discussed in detail. The mathematical treatment is kept on an elementary level throughout.

A. E. Heins (Cambridge, Mass.).

Verschaffelt, J.-E. The thermomechanics of the electric conductor. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 631–649 (1941). (Flemish) [MF 12676]

Gray, Marion C. A modification of Hallén's solution of the antenna problem. J. Appl. Phys. **15**, 61–65 (1944). [MF 12496]

Hallén's theory of a thin cylindrical antenna [Nova Acta Soc. Sci. Upsaliensis (4) **11**, 1–44 (1938)] leads to the integral equation

$$\int_{-l}^{l} I(\xi)e^{-i\beta r}r^{-1}d\xi=(4\pi/\eta i)\{A\cos\beta z+V_0\sin\beta|z|\},$$

in which $I(\xi)$ is the unknown current distribution in the antenna, $r^2=(z-\xi)^2+a^2$, and a, A, β, η, l, V_0 are constants, all known except A which must be determined so as to make $I(l)=0$. In particular, a is the (small) radius of the antenna. Hallén expanded the solution in a series of powers of the parameter $\{2\log(2l/a)\}^{-1}$. The author gives an improved approximate solution. She introduces the function

$$Z(z)=(\tfrac{1}{2}\eta/\pi)\int_{-l}^{l}r^{-1}\cos\beta r d\xi,$$

rewrites the integral equation in the form

$$I(z)Z(z)=-2i(A\cos\beta z+V_0\sin\beta|z|)$$
$$-(\tfrac{1}{2}\eta/\pi)\int_{-l}^{l}\{I(\xi)-I(z)\}r^{-1}\cos\beta r d\xi$$
$$+(\tfrac{1}{2}i\eta/\pi)\int_{-l}^{l}I(\xi)r^{-1}\sin\beta r d\xi,$$

puts $a=0$, and then solves the integral equation by successive approximations. The first approximation is

$$I_0(z)=-2i\{A\cos\beta z+V_0\sin\beta|z|\}/Z(z).$$

The second approximation leads to integrals of the type

$$\int\frac{\cos\beta|z-\xi|}{|z-\xi|}\left\{\frac{\cos\beta\xi}{Z(\xi)}-\frac{\cos\beta z}{Z(z)}\right\}d\xi,$$

in which the slowly varying function $Z(x)$ may be replaced by a constant average value $\bar{Z}$, whereupon the integrals can be evaluated in terms of cosine and sine integrals. Numerical evaluation shows a marked deviation from Hallén's results but close agreement with results of Schelkunoff [J. Appl. Phys. **15**, 54–60 (1944)] and other writers.

A. Erdélyi (Edinburgh).

Wang, C. C. Electromagnetic field inside a cylinder with a gap. J. Appl. Phys. **16**, 351–366 (1945). [MF 12541]

The author investigates the field in two semi-infinite metallic circular cylinders of common axis and equal inner radius a; the cylinders are separated by a distance d. In cylindrical coordinates, the axial component of the electric field of circular symmetry will be of the form

$$(1)\qquad E_z(r,z)=\int_0^\infty A(h)J_0(r(k^2-h^2)^{\frac{1}{2}})\cos hz dh.$$

By Fourier's inversion theorem,

$$A(h)J_0(a(k^2-h^2)^{\frac{1}{2}})=\pi^{-1}\int_{-\infty}^{\infty}E_z(a,z)\cos hz dz.$$

On the metallic cylinders, that is, for $\pm z>\frac{1}{2}d$, $E_z(a,z)=0$; if the excitation is such as to produce a uniform field at the gap, that is, $E_z(a,z)=E_a=$constant for $-\frac{1}{2}d<z<\frac{1}{2}d$, then

$$A(h)=\frac{2E_a\sin\frac{1}{2}hd}{\pi J_0(a(k^2-h^2)^{\frac{1}{2}})}.$$

With this value of $A(h)$, (1) is evaluated by the calculus of residues under the assumption that $ka<2.405$, that is, that all poles of the integrand lie on the imaginary axis of the u-plane. The originally slow convergence of the resulting series is improved in the well-known manner by subtracting a known series the terms of which approximate asymptotically to its terms. A corresponding representation is given for the radial electric field. Curves are calculated which can be used to estimate the relative field strength at different points inside the cylinder. Equivalent capacitance due to the energy storage in the cylinder is graphed; it can be used for estimating the resonant frequency of the associated cavity. The limiting form $k\to 0$ (electrostatic field) of the formulae is obtained; this field can be used for electron optical calculations. *A. Erdélyi* (Edinburgh).

Liebowitz, Benjamin. Development of electromagnetic theory for non-homogeneous spaces—a correction. Phys. Rev. (2) **67**, 364–365 (1945). [MF 12639]

The paper referred to appeared in Phys. Rev. (2) **64**, 294–301 (1943); these Rev. **5**, 219.

Livens, G. H. On the energy and mechanical relations of the magnetic field. Philos. Mag. (7) **36**, 1–20 (1945). [MF 12685]

The relations are discussed by means of Lagrange's method in dynamics. It is concluded that the simplest and most natural form of a theory of magnetism is one in which the vector B is taken as the fundamental force vector in the field, in the sense in which this term is usually applied; H is then the induction, although now it is the induction of mechanically effective force that is implied. The aim is to overcome some of the difficulties which occur when $\mu H^2/8\pi$ is used for the density of the distribution of energy and the total energy defined by this distribution is replaced by the familiar expression

$$(1/2c)\sum(L_{ss}i_s^2+2L_{rs}i_ri_s)$$

in terms of the currents i_s and the coefficients L_{ss}, L_{rs} of self and mutual induction of the circuits. In particular, there is the anomaly that, when a current and a magnet exist together in the same field, the force on the magnet exerted by the current tends to increase the energy, while the equal and opposite reactions on the current would tend to decrease it. One point of interest is that the density $(1/8\pi)\{\mu H^2-2(BH)\}$ gives exactly the same total energy as the original formula for the distribution of energy density. *H. Bateman.*

Podolsky, Boris, and Kikuchi, Chihiro. Auxiliary conditions and electrostatic interaction in generalized quantum electrodynamics. Phys. Rev. (2) **67**, 184–192 (1945). [MF 12216]

The paper is a continuation of the authors' generalized electrodynamics [Phys. Rev. (2) **62**, 68–71 (1942); **65**, 228–235 (1944); these Rev. **4**, 31; **5**, 277], in which the term $\partial F_{\alpha\beta}/\partial x_\beta$ of Maxwell's equations is replaced by

$$(1-a^2\Box)\partial F_{\alpha\beta}/\partial x_\beta$$

in order to obtain a modification of the field intensities accounting for short-range forces (of order $\exp(-r/a)$) of the Yukawa type, besides the Coulomb-Lorentz force. The auxiliary condition $\partial\phi_\alpha/\partial x_\alpha=0$, to which the potentials are subjected in Maxwell's theory, is replaced by the condition $a\cdot\partial\phi_\alpha/\partial x_\alpha+B=0$, where $(1-a^2\Box)B=0$. Such auxiliary conditions present some difficulties in the quantization of electrodynamics, which the authors have set up by extending the procedure of Heisenberg and Pauli [Z. Phys. **56**, 1–61 1929)]. In the paper under review, the authors deal with these difficulties along the lines of Fock [Phys. Z. Sowjetunion **6**, 425–469 (1934)] and then deduce the relativistic wave equation for charged particles in the generalized field. Their result differs from the corresponding result for a Maxwellian field in the appearance of a short-range interaction term between the charged particles, besides the Coulomb interaction, and of an electrostatic self-energy term like that of a charged sphere of radius a, which the authors had already derived in their classical theory. The result also differs in that the vectors, D say, which represent the interaction between the various particles and the field, have nonvanishing divergence and, instead, satisfy the equation $(1-a^2\Box)\,\mathrm{div}\,D=0$. *P. Weiss* (London).

Powell, F. C. The dynamics of a simple system. I. Observables. Proc. Cambridge Philos. Soc. **41**, 57–65 (1945). [MF 12376]

Since not all the axioms of the theory of Hilbert space have physical significance in quantum mechanics, it is desirable to exhibit a set of physical postulates from which it would "follow (i) that the observables have the properties assumed in quantum mechanics and (ii) that the equations of motion are of quantum mechanical form." An attempt is made to do this in the case of finite systems. Accordingly, the author introduces a collection of at least 9 postulates, 9 definitions, and 13 short theorems. The exposition is such that an exact count is impracticable. Unfortunately, part of the paper is obscure. For instance, the reviewer was unable to supply the omitted proof of the fundamental relation

$$H\times(A\times B)=(H\times A)\times B+A\times(H\times B)$$

on the basis of the author's definition of the "skew" product $A\times B$. Moreover, the appeal to classical mechanics (in paragraph 7.1) in an attempt to motivate the introduction of one of the principal postulates is unsatisfactory. It is there stated that "the first order equations of motion express the rate of change of a dynamical variable A in terms of the Hamiltonian and the variable conjugate to A," but that "the second order equations express the second rate of change of A in terms of the Hamiltonian and A itself, without reference to the conjugate variable." In general, of course, the rate of change of the variable p_i is given in classical mechanics by $-\partial H/\partial q_i$ and the second rate of change is given by

$$\sum_{j=1}^{n}\left(\frac{\partial^2 H}{\partial p_j\partial q_i}\frac{\partial H}{\partial q_j}-\frac{\partial^2 H}{\partial q_j\partial q_i}\frac{\partial H}{\partial p_j}\right).$$

It is hard to see in what sense the latter expression can be expressed in terms of p_i and H that would not apply with equal force to the former expression. *D. C. Lewis.*

Mandeljštam, L. I., and Tamm, I. E. Reciprocity of energy-time indeterminacy in nonrelativistic quantum mechanics. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **9**, 122–128 (1945). (Russian) [MF 12885]

Yvon, Jacques. Équations de Madelung de l'électron magnétique non relativiste. Revue Sci. (Rev. Rose Illus.) **79**, 209–215 (1941). [MF 12862]

Gårding, Lars. Relativistic wave equations for zero rest-mass. Proc. Cambridge Philos. Soc. **41**, 49–56 (1945). [MF 12375]

Let us assume that we have a spinor $u^{\alpha_1\cdots\alpha_m\dot\beta_1\cdots\dot\beta_n}$, where m and n are arbitrary integers. We assume that each component of such a spinor satisfies the equation $\Box u=0$. The question then is: What is the relativistically invariant minimum set of linear partial homogeneous differential equations leading to the wave equations? The general discussion is fairly complicated because many possibilities, for different m and n, must be considered in each case. The equations are of the type

$$s\dot s p^{\alpha\dot\beta}u^{\alpha_1\cdots\alpha_m\dot\beta_1\cdots\dot\beta_n}=0,$$

where $p^{i\dot i}=\partial/\partial x^1+\partial/\partial x^2$, etc.; s ($\dot s$), the symmetrizing operator, operates on undotted (dotted) indices. The special and relatively simple cases of Maxwell's and Dirac's equations with mass zero are regained at the end of the paper. The

more general case, starting from $\Box u = au$ (a constant), is treated briefly in a note. *L. Infeld* (Toronto, Ont.).

Majumdar, R. C. The quantum theory of radiation damping. Proc. 32nd Indian Sci. Congress, Nagpur 1945, part II, 28 pp. (1945). [MF 12615]

Presidential address reviewing recent progress in the subject. There is a bibliography containing 78 items.

Kar, K. C. The distortion of plane χ-wave and its effect on elastic scattering in Coulomb field. Indian J. Phys. 18, 144–147 (1944). [MF 12540]

This note concerns an attempt to improve the wave-mechanical theory of the elastic scattering of particles by a Coulomb field of force. The fundamental problem is that of obtaining a function which satisfies the Schrödinger equation

$$\nabla^2\psi + k^2[1-(Ze^2)/(Er)]\psi = 0,$$

and which has appropriate behavior for large values of r. The common procedure is to employ a perturbation method, the basic unperturbed approximation being taken to be a plane wave. In this note, however, the basic unperturbed approximation is taken to be a function of the form $\exp i\{kx+\alpha \log k(r-x)\}$. Such a function is considered to be a better representation of the incident wave than the customary function $\exp(ikx)$. The perturbation method is then continued in the usual way. It turns out that this refinement in the analysis does not result in any physically significant changes in the final formulae. *L. A. MacColl.*

Chakrabarty, S. K. Photons associated with a cascade shower. Bull. Calcutta Math. Soc. 36, 135–140 (1944). [MF 12412]

An expression is found for the number $Q(E, t)dE$ of photons with energy between E and $E+dE$ at a depth t, in radiation units, below the surface of the material in which the shower is produced, by taking ionization loss into consideration as well as the other processes considered previously [Bhabha and Chakrabarty, Proc. Indian Acad. Sci., Sect. A. 15, 464–476 (1942); Chakrabarty, Proc. Nat. Inst. Sci. India 8, 331–337 (1942); these Rev. 4, 236]. Mellin's transformation is applied to the cascade equations and, after the resulting differential-difference equations have been solved with the proper boundary conditions, $Q(E, t)$ is found by the inverse of Mellin's transformation. The results are given in 3 tables. *H. Bateman.*

Hulthén, Lamek. Über die Eigenlösungen der Schrödinger-Gleichung des Deuterons. Ark. Mat. Astr. Fys. 28A, no. 5, 12 pp. (1942). [MF 11544]

According to the meson theory of nuclear forces the radial Schrödinger equation of the deuteron has the form

$$(1) \qquad [A(d^2/dr^2 - l(l+1)/r^2) + Be^{-\gamma r}/r + E]R = 0.$$

This can be reduced by a change of variables to the form (2) $\phi'' + (a+be^{-x}/x)\phi = 0$. Equation (2) has been handled previously by numerical methods or by the differential analyzer. The author shows that (2) can be treated analytically. He first considers the equation

$$(3) \qquad \psi'' + (a+be^{-x}/(1-e^{-x}))\psi = 0.$$

This approximation to (2) is reduced by a transformation of variables to the Jacobi equation

$$(4) \qquad x(x-1)\chi'' - \alpha x\chi' + b\chi = 0,$$

whose theory is well known. Equation (2) is then studied by perturbation methods applied to (4). In this way the author finds approximate formulas for the eigenfunctions and approximate values for the eigenvalues of (2), which he compares with those obtained by others by numerical integration. The results throw some light on the physical questions of the probable mass of the meson and the probable values of certain constants connected with meson spin which occur in the coefficient B of equation (1).

O. Frink (Xenia, Ohio).

Jauch, J. M., and Lopes, J. Leite. Scalar meson pair theory of nuclear forces. Anais Acad. Brasil. Ci. 16, 281–289 (1944). [MF 12211]

All meson theories of nuclear forces which are symmetrical in the charge and in which the interaction between the meson and the nucleon is linear in the meson field operators seem to lead to a wrong angular dependence for the scattering of fast neutrons by protons. The authors investigate the possibility that the interaction term is quadratic in the field operators. This corresponds physically to the case where the nucleons emit and absorb mesons in pairs, and not singly as in the linear theories. Considering mesons of spin 0 (scalar wave function) and using only the wave field and its first derivatives, the authors point out that there are only two possibilities, one spin independent and the other spin dependent, for representing the interaction in the Lagrangian function. The calculation of the interaction potential between nucleons by means of the weak coupling perturbation method shows that the former gives a spin independent attractive force with a permissible singularity at the origin of type $1/r$, the latter, a spin dependent force containing the tensor interaction, with singularity of type r^{-3}. An examination of the sign of the spin dependent potential shows that it is repulsive in the triplet state so that the only possibility is the superposition of the two interaction terms. An estimate of the two coupling constants by requiring that the deuteron have the observed values of the binding energy and the quadrupole moment gives such large values to the interaction as to violate the weak coupling condition assumed initially in the calculation. *S. Kusaka* (Northampton, Mass.)

Hu, Ning. The relativistic correction in the meson theory of nuclear force. Phys. Rev. (2) 67, 339–346 (1945). [MF 12659]

BIBLIOGRAPHICAL NOTES

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